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FOUNDATIONS

*Hilbert, D. und Bernays, P. **Grundlagen der Mathematik. Vol. I.** J. W. Edwards, Ann Arbor, Michigan, 1944. xii+471 pp. $12.40.

Reprint of volume 50 of the series "Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen" [J. Springer, Berlin, 1934].

*Hilbert, D. und Bernays, P. **Grundlagen der Mathematik. Vol. II.** J. W. Edwards, Ann Arbor, Michigan, 1944. xii+498 pp. $13.20.

Reprint of volume 51 of the series "Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen" [J. Springer, Berlin, 1939].

*Church, Alonzo. **Introduction to Mathematical Logic. I.** Annals of Mathematics Studies, no. 13. Princeton University Press, Princeton, N. J., 1944. vi+118 pp. $1.75.

The content of this monograph is approximately "the first half of an introductory course in mathematical logic given to graduate students in mathematics" at Princeton in 1943. It is written with the meticulous precision which characterizes the author's work generally. The treatment is formally deductive with relatively little explanatory matter; as the author says himself, the tract is not suited as an introduction for students with little mathematical background. The subject matter is more or less classical, namely, the propositional algebra and the functional calculus of first order, to which is added a chapter summarizing without proofs certain features of functional calculi of higher order. For the expert the chief interest in the tract is that it makes readily accessible careful detailed formulation and proofs of certain standard theorems, for example, the deduction theorem, the reduction to truth tables, the substitution rule for the functional calculus, Gödel's completeness theorem, etc. It is hoped that a second volume will follow.

H. B. Curry (Silver Spring, Md.).

Schröter, Karl. Ein allgemeiner Kalkülbegriff. Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften (N.F.) **6**, 43 pp. (1941). [MF 10464]

The author presents a new axiomatic development of metamathematics. In the first chapter an axiom system is given for the theory of concatenation. What the author calls a "semiotic quadruple" is a quadruple whose first member is a thing (to be interpreted as the "empty" sign), whose second member is a class (to be interpreted as the class of atomic signs), whose third member is also a class (to be interpreted as the class of expressions) and whose fourth member is a three-place predicate (to be interpreted so as to be true of three things x, y and z when x results from y and z by concatenation). Semiotic quadruples are defined in an axiomatic way, and it is shown that the axioms are independent. Examples are also given to show that for every cardinal number α there exists a semiotic quadruple whose class of atoms contains just α members. It is shown that a necessary and sufficient condition for a semiotic quadruple a to be a homomorphic image of a semiotic quadruple b is that the number of atoms in a be less than or equal to the number of atoms in b. Hence it follows that a necessary and sufficient condition that a semiotic quadruple a be isomorphic to a semiotic quadruple b is that a and b have the same number of atoms.

In the second chapter the theory of the first chapter is extended by including also the notion of logical consequence. Axioms are given for this notion, and the axioms are shown to be independent. Various relations between two realizations of these axioms are defined, for example, the relation of isomorphism and two different relations of inclusion of one realization in another.

This paper leans heavily on the work of Tarski [especially Studia Philos. (Comment. Soc. Philos. Polon.) **1**, 261–405 (1935); Monatsh. Math. Phys. **37**, 361–404 (1930)]. The chief new results here seem to be: (1) the condition for isomorphism of two semiotic quadruples and (2) the combination of the theory of concatenation with the theory of logical consequence.

J. C. C. McKinsey.

McKinsey, J. C. C. On the number of complete extensions of the Lewis systems of sentential calculus. J. Symbolic Logic **9**, 42–45 (1944).

It is proved that there are infinitely many complete extensions of the Lewis systems of strict implication S1 and S2, but that there are only unique complete extensions of S4 and S5. A system S (of sentential calculus) is an extension of a system R if R and S have the same class of significant formulae and if every provable formula of R is also a provable formula of S; the system is a proper extension if in addition not all provable formulae of S are provable formulae of R. A system is complete if it is consistent but has no consistent proper extensions. The proof for S4 (and S5) follows from a general principle due to Tarski, whereas that for S2 (and S1) follows essentially from a result of Lindenbaum. The question as to how many complete extensions there are for S3 is not yet solved.

R. M. Martin.

Post, Emil L. Recursively enumerable sets of positive integers and their decision problems. Bull. Amer. Math. Soc. **50**, 284–316 (1944). [MF 10496]

This paper contains an intuitive presentation of a certain portion of the theory of (general) recursive functions of positive integers. The definition of recursive function (due to Herbrand, Gödel and Kleene) is presupposed; a recursively enumerable set of positive integers is then defined as the set of values of a general recursive function. Just as, following Church, the notion of general recursive function may be regarded as the exact equivalent of the intuitive notion of effectively calculable function, so too the notion

of a recursively enumerable set may be regarded as the exact equivalent of the intuitive notion of an effectively calculable set. The formalism worked with here is that of a previous paper [Amer. Math. J. 65, 197–215 (1943); these Rev. 4, 209] and certain concepts of that paper are presupposed, in particular, that of a system in normal form. Each system in normal form may be thought of as defining a set (possibly null) of positive integers. Hence every recursively enumerable set of positive integers is the set of integers defined by some normal system and conversely. By the decision problem for a given set of positive integers is understood the problem of effectively determining whether an arbitrarily given positive integer is or is not a member of the set. Now the decision problem of an arbitrary logistic system may be transformed into the decision problem of a certain (recursively enumerable) set of positive integers, namely, some set which has effectively been put into one-to-one correspondence with the set of asserted formulae of the system. The decision problem is recursively solvable or unsolvable, namely, according as the corresponding set S is or is not recursive, that is, according as there does or does not exist a recursive function $f(x)$ such that $f(x)=2$ if $x \varepsilon S$, and $f(x)=1$ if $x \bar{\varepsilon} S$.

It is impossible in a short review to do justice to the various and interesting exposés of this paper, many of which are new. Post first cites several theorems on the relationship between recursive and recursively enumerable sets, including the theorem that a set of positive integers is recursive if and only if both it and its complement with respect to the set of all positive integers are recursively enumerable, and the theorem (due essentially to Church) that a recursively enumerable set of positive integers exists which is not recursive. Also a form, close to that of Kleene [Math. Ann. 112, 727–742 (1936), in particular, p. 740, XIII], of the Gödel incompletability theorem is proved.

Certain special kinds of sets of positive integers are defined, namely, creative, simple and hyper-simple sets. Each such set is recursively enumerable but not recursive, and the decision problem of each such set is recursively unsolvable. The decision problem of a set of integers S_1 is said to be many-one reducible to that of S_2 if we are in possession of a recursive method which would determine for each n an m such that n is or is not in S_1 according as m is or is not in S_2; it is one-one reducible if distinct n's lead to distinct m's. It follows that the decision problem of every recursively enumerable set of positive integers is one-one reducible to the decision problem of a certain (very important) creative set K. Finally, for unsolvable problems the concept of reducibility gives way to the concept of comparative degree of unsolvability. An unsolvable problem is of the same degree as a second if each is reducible to the other; it is of lower degree than a second if it is reducible to the second but the second is not reducible to it. The question as to whether there exist recursively enumerable sets of positive integers of lower degree of unsolvability than the set K, or whether all recursively enumerable sets of positive integers with recursively unsolvable decision problems are of the same degree of unsolvability, is left open.

R. M. Martin (Princeton, N. J.).

Goodstein, R. L. On the restricted ordinal theorem. J. Symbolic Logic 9, 33–41 (1944).

By the "restricted ordinal theorem," the author means the proposition that a descending sequence of ordinals, each less than ϵ (the first ordinal to satisfy $\epsilon=\omega^{\epsilon}$), is necessarily finite. This theorem is of importance because of the crucial role it plays in Gentzen's proof of the consistency of his "reine Zahlentheorie." In view of the results of Gödel, the restricted ordinal theorem cannot be proved in Gentzen's system; for, if it could, Gentzen's proof of consistency could be formalized within the system.

The author formulates an interesting purely number-theoretic proposition which is equivalent to the restricted ordinal theorem. He then presents a finitist proof (in part due to Bernays) of the restricted ordinal theorem for ordinals up to $\omega^{\omega^{\omega}}$. He remarks that his method of proof can be extended to larger ordinals, though apparently the details would become increasingly complicated. He ventures the guess, however, "that a more subtle approach would enable the theorems to be proved, by finitist methods, for ordinals up to any assigned v_n, where $v_0=\omega$, $v_{n+1}=\omega^{v_n}$."

J. C. C. McKinsey (Bozeman, Mont.).

van Dantzig, D. On the affirmative content of Peano's theorem on differential equations. Nederl. Akad. Wetensch., Proc. 45, 367–373 (1942). [MF 10404]

According to Peano [1890] the fundamental problem for differential equations

$$(F) \qquad dx/dt=f(t, x(t)), \qquad x(0)=0,$$

has at least one solution (satisfying $|x(t)|\leqq 1$ for $0\leqq t\leqq 1$), provided f is uniformly continuous and $|f(t, x)|\leqq\alpha<1$ for $0\leqq t\leqq 1$, $|x|\leqq 1$. The proof consists of two parts, the first being of finite constructive character and approximating by way of imbedding in the totality of all solutions, the second picking out one by dint of the principle that the point set on which a nest of closed subsets $S_1>S_2>\cdots$ of a compact set S closes in is not void. The paper contains an explicit formulation of the first "affirmative" part, acceptable to the strictest intuitionist.

Applying the difference method, one divides the interval $0\leqq t\leqq 1$ into equal parts J of length 2^{-l}. From this dyadic partition P_l of rank l one passes to P_{l+1} by breaking each J into two equal halves J', J''. Let us define to be a P_l-graph any continuous function $x(t)$ which is linear in each of the 2^l intervals J of P_l; we add the condition $x(0)=0$. Such a graph may be characterized by its gradients m_J. An admissible function $x(t)$, that is, one for which $x(0)=0$ and $|\Delta x/\Delta t|\leqq 1$ $(0\leqq t,\ t+\Delta t\leqq 1)$ may be characterized by the difference quotients $m_J[x]=\Delta_J x/\Delta_J t$ corresponding to the dyadic intervals J of rank $l=0, 1, 2, \cdots$. They satisfy the conditions $|m_J|\leqq 1$, $m_J=\frac{1}{2}(m_{J'}+m_{J''})$. Conversely, any given set of numbers m_J satisfying these conditions defines a corresponding admissible function $x(t)$, $m_J[x]=m_J$, and hence may serve as set of coordinates in the functional space S of admissible functions.

Denoting by β a suitable chosen universal constant, suppose that $f(t, x)$ is fixed with an error not greater than $\beta\cdot 2^{-k}$ as soon as t and x are known with an error not greater than 2^{-l}, $l=l(k)$. If $x(t)$ is a solution of (F) it varies by less than $\alpha\cdot 2^{-l}$ while t ranges over one of the dyadic intervals J of rank l, and hence $f(t, x(t))$ fluctuates by less than $\beta\cdot 2^{-k}$. Let $\xi(t)$ be any P_l-graph the gradients of which are multiples of 2^{-k} and lie between -1 and $+1$, and let each be surrounded in S by the block $|m_J-m_J[\xi]|\leqq 2^{-k}$, the inequalities holding simultaneously for all dyadic intervals J of rank $0, 1, 2, \cdots$. There is only a finite number of such blocks. A number of them, namely, those whose centers ξ satisfy a certain difference inequality paralleling the differential equation (F), are joined to form the set S_k. The main

The proof consists of two parts; the first, which is of finite constructive character, approximates the totality of all solutions by an embedding process while the second picks out one solution according to the principle that ...

result consists of the two complementary statements: (1) any admissible function $x(t)$ satisfying the inequality

$$(F_k) \quad \left|\Delta x - \int_t^{t+\Delta t} f(t, x(t))dt\right| \leqq \beta 2^{-k'} \cdot |\Delta t|, \quad 0 \leqq t,\ t+\Delta t \leqq 1,$$

lies in S_k; (2) conversely, if $k'=k'(k)>k$ is a certain integer depending on k in an elementary manner described explicitly, then $x(t) \epsilon S_{k'}$ implies (F_k). All this holds from a sufficiently high integer k on.

The author replaces the coordinates m_J bound to the relations $m_J=\frac{1}{2}(m_{J'}+m_{J''})$ by independent quantities c_J in terms of which the m_J are linearly expressed. The P_l-graph which approximates $x(t)$ by coinciding with it at the points $j \cdot 2^{-l}$ $(j=0, 1, \cdots, 2^l)$ of P_l is changed into the approximating P_{l+1}-graph by adding a P_{l+1}-graph (zig-zag) which equals 0 at the points of P_l and has a certain value c_J at the middle of each interval J of P_l. Put $x(1)=c$. Then the m_J for the dyadic intervals J of rank $l+1$ are linearly expressed in terms of c and the c_J corresponding to the dyadic intervals J of rank $0, 1, \cdots, l$. The intervals J of all ranks $0, 1, 2, \cdots$ can be enumerated by an index $n=1, 2, \cdots$. The resulting formulas in terms of the coordinates c_n look somewhat complicated. The argument is carried through not only for a single differential equation, but also for a set, where x and f denote vectors with ν components $x_1, \cdots, x_\nu$; $f_1, \cdots, f_\nu$.

[Something is wrong with the example given at the beginning of the paper.] *H. Weyl* (Princeton, N, J.).

van Dantzig, D. A remark and a problem concerning the intuitionistic form of Cantor's intersection theorem. Nederl. Akad. Wetensch., Proc. **45**, 374–375 (1942). [MF 10405]

Let $S_1 > S_2 > \cdots$ be a nest of subsets of the interval (01), each member of which is the join of some closed dyadic intervals. The statement that there are points common to all S_n is not universally true from the intuitionistic standpoint; nor may one expect that for an example constructed explicitly the existence of a common point is demonstrably absurd. A "weaker" interpretation of both the affirmative and the negative statement is suggested, the validity of which, however, the author leaves undecided. Attention is called to these difficulties, because they are characteristic for the propositions and pretensions of classical analysis.

H. Weyl (Princeton, N. J.).

Brouwer, L. E. J. Zum freien Werden von Mengen und Funktionen. Nederl. Akad. Wetensch., Proc. **45**, 322–323 (1942). [MF 10396]

Brouwer, L. E. J. Beweis dass der Begriff der Menge höherer Ordnung nicht als Grundbegriff der intuitionistischen Mathematik in Betracht kommt. Nederl. Akad. Wetensch., Proc. **45**, 791–793 (1942). [MF 10435]

The first note points out not only that individual continuous functions play a role in mathematics, but that there are many possibilities of defining a functional space such that the individual function of that space is determined by a law, but its "arbitrary" function depends on a sequence of free choices. Thus the power series $\sum \epsilon_n x^n/n!$ with $\epsilon_n = \pm 1$ form a functional space in which an arbitrary function is in the process of growing if one freely chooses one after the other of the signs $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$. A number of remarks clarifying some points in the fundamental definitions of intuitionistic mathematics are added.

The following question is raised. "Zur Erklärung einer Menge M gehört nach der Mengendefinition eine Fundamentalreihe $\alpha(M)$ von Zeichenreihen, welche der Fundamentalreihe der in einer bestimmten Weise abgezählten endlichen Wahlfolgen eineindeutig zugeordnet ist, und zwar ist nach der Mengendefinition dieses $\alpha(M)$ für M von vornherein festgelegt. Man könnte nun die Frage aufwerfen, ob nicht auch die Betrachtung "schwebender Mengen" M_σ, für welche die entsprechenden $\alpha(M_\sigma)$ sich im "freien Werden" befinden, mathematische Fruchtbarkeit besässe. In den Fällen, für welche diese Frage bejahend zu beantworten wäre, werden sich meiner Ueberzeugung nach die betreffenden M_σ immer als mathematische Entitäten oder Spezies herausstellen. Ein einfacher derartiger Fall tritt, zum Beispiel, ein für diejenigen M_σ, deren $\alpha(M_\sigma)$ die Elemente einer gegebenen Menge M darstellen. Diese M_σ sind Teilspezies einer aus M herleitbaren Menge M_1, mit welcher ihre Vereinigung identisch ist."

The second note proves indeed that such a "Menge M_σ of second order" is a well defined subspecies of a Menge M_1 derived from M by an elementary process. Mengen of even higher order than the second add nothing new.

H. Weyl (Princeton, N. J.).

Brouwer, L. E. J. Die repräsentierende Menge der stetigen Funktionen des Einheitskontinuums. Nederl. Akad. Wetensch., Proc. **45**, 443 (1942). [MF 10408]

In intuitionistic analysis there is a narrower concept of continuous function ("Blockfunktion") based on the correspondence of intervals for argument x and value y, and a wider one in which the shrinking y-interval is determined by a "Punktkern" of the argument (a Punktkern being the species of all points coinciding with a given point x in the unit interval $0 \leqq x \leqq 1$). It is the latter concept for which the author here introduces the term "volle Freifunktion" and to which he generalizes his former proof that a "volle Funktion" is uniformly continuous [Math. Ann. **97**, 60–70 (1926), in particular, pp. 66–67]. It then follows that each volle Freifunktion may be represented by a "Blockfunktion."

H. Weyl (Princeton, N. J.).

Tarski, Alfred. The semantic conception of truth and the foundations of semantics. Philos. and Phenomenol. Res. **4**, 341–376 (1944). [MF 10647]

This paper contains an extremely clear, popular account of the main results of Tarski's researches upon the concept of truth and upon the foundations of semantics [see especially Studia Philosophica **1**, 261–405 (1930)]. The first part of the paper is expository, whereas the second answers certain of the (for the most part philosophical) objections that have been urged against Tarski's theory.

An adequate definition of truth must be both materially adequate and formally correct. Consider an arbitrary sentence. Replace it by the letter "p." Form the name of this sentence and replace it by the letter "X." We certainly then have the following equivalence: X is true if and only if p. The term "true" is adequate materially if all equivalences of this form can be asserted. Formal correctness is to be obtained only by specifying in a rigorous way the formal structure of the language in which the definition is to be given. Philosophically the semantic conception of truth should, Tarski thinks, conform essentially to the dictum of Aristotle that to say of what is that it is not or of what is not that it is, is false, whereas to say of what is that it is or of what is not that it is not, is true.

The definition Tarski has proposed presupposes the fundamental semantic relation (having objects and expressions as relata) of satisfaction, a general definition of which may be given recursively. A sentence then is said to be true if it is satisfied by all objects, and false otherwise. To define recursively, however, the notion of satisfaction requires a certain form of definition not admitted in the particular object-language, and hence can be given only in a metalanguage essentially richer as to its logical content.

R. M. Martin (Princeton, N. J.).

*von Wright, Georg Henrik. **The Logical Problem of Induction.** Acta Philosophica Fennica, Fasc. 3, 1941. Societas Philosophica, Helsinki, 1941. 258 pp.

Broad, C. D. Hr. von Wright on the logic of induction. I. Mind **53**, 1–24 (1944). [MF 10960]

Broad, C. D. Hr. von Wright on the logic of induction. II. Mind **53**, 97–119 (1944). [MF 10961]

Broad, C. D. Hr. von Wright on the logic of induction. III. Mind **53**, 193–214 (1944). [MF 10962]

Cf. the title quoted above.

Wilder, R. L. The nature of mathematical proof. Amer. Math. Monthly **51**, 309–323 (1944). [MF 10677]

This paper has the character of a "causerie"; it consists of a nontechnical discussion, of value to mathematicians as well as to interested laymen, of some of the recent ideas concerning mathematical proof. It is not primarily concerned with the logical and philosophical aspects of these ideas, but rather with the preconceptions, sometimes not consciously avowed, which underlie and unavoidably influence the point of view of active mathematicians. The last four pages contain an appendix, intended to introduce the reader to the concepts with which the paper deals.

A. Dresden (Swarthmore, Pa.).

Fraenkel, Abraham Adolf. Problems and methods in modern mathematics. Scripta Math. **9**, 245–255 (1943). [MF 11035]

Ore, Oystein. Mathematics for students of the humanities. Amer. Math. Monthly **51**, 453–458 (1944). [MF 11247]

Webb, D. A. The place of mathematics in scientific method. Hermathena **64**, 64–87 (1944). [MF 11448]

Sen, N. R. The concept of space in mathematics and natural philosophy. Bull. Calcutta Math. Soc. **36**, 1–6 (1944). [MF 11210]

Whittaker, E. T. The new algebras and their significance for physics and philosophy. Philos. Mag. (7) **35**, 1–15 (1944). [MF 10862]

This paper is identical with the author's paper in the Year Book, Roy. Soc. Edinburgh **1944**; these Rev. **5**, 86.

*Servien, Pius. **Base physique et base mathématique de la théorie des probabilités. Vers une nouvelle forme de la théorie.** Actual. Sci. Ind., no. 908. Hermann et Cie., Paris, 1942. 55 pp.

This volume belongs to the series "Esthétique" of which the author is editor.

ALGEBRA

v. Schrutka, Lothar. Eine neue Einteilung der Permutationen. Math. Ann. **118**, 246–250 (1941). [MF 10704]

The "new" classification of permutations of the elements $1, 2, \cdots, n$ is according to runs (maximal subsequences of increasing numbers). The author gives recursion formulas for the number of permutations with k runs, and representations by means of higher order differences and by coefficients in certain power series. Classical papers and textbooks are quoted for all these results. *W. Feller.*

Shyü, Kintzyur. Two theorems concerning combinations. Duke Math. J. **11**, 293–299 (1944). [MF 10668]

The theorems are rather complicated; it will suffice to state the second as follows. Let $k_1, \cdots, k_n$ be integers greater than 0; then

$$\sum_{s} \prod_{\nu=1}^{n}\left[a\binom{x_\nu}{k_1}+b\binom{x_\nu}{k_2}+\cdots\right] = \sum \frac{n!}{\alpha!\beta!\cdots}\binom{m+n-1}{\alpha k_1+\beta k_2+\cdots+n-1}a^\alpha b^\beta\cdots,$$

where the sum on the left is over all $x_\nu \geqq 1$ such that $x_1+\cdots+x_n=m$, and the sum on the right is over all $\alpha, \beta, \cdots \geqq 0$, such that $\alpha+\beta+\cdots=n$. Various consequences of the theorems are derived. *L. Carlitz* (Durham, N. C.).

Aitken, A. C. On the number of distinct terms in the expansion of symmetric and skew determinants. Edinburgh Math. Notes no. **34**, 1–5 (1944). [MF 11301]

The numbers referred to in the title (first studied by Cayley) are determined in an elementary way, using recursion formulas. *W. Feller* (Providence, R. I.).

Puri, Amritsagar. On some problems in the theory of equations. Math. Student **11**, 15–20 (1943). [MF 10993]

Courtois, Jacques. Produits symétriques. C. R. Acad. Sci. Paris **215**, 570–572 (1942). [MF 10190]

This brief note mentions several immediate properties (for example, commutativity, associativity) of a rather obscurely formulated notion of "symmetric product" of a set of functions. If $r_1, \cdots, r_N$ are N systems of variables and $\phi_1, \cdots, \phi_p$ $(p \leqq N)$ are p functions each defined for any r_i, then the symmetric product of $\phi_1, \cdots, \phi_p$ is taken as the arithmetic mean of all products

$$\Pi\phi(r_{i_1})\phi(r_{i_2})\cdots\phi(r_{i_N}),$$

where $i_1, i_2, \cdots, i_N$ is a permutation of $1, 2, \cdots, N$.

D. H. Lehmer (Berkeley, Calif.).

Perron, Oskar. Beweis und Verschärfung eines Satzes von Kronecker. Math. Ann. **118**, 441–448 (1942). [MF 10708]

The author proves the theorem: given n homogeneous polynomials of the mth degree in k variables with coefficients in a field of characteristic zero:

$$f_s(x_1, x_2, \cdots, x_k), \qquad s=1, 2, \cdots, n,$$

it is always possible to find k linear combinations

$$g_r(x_1, x_2, \cdots, x_k)=\sum_{s=1}^{n} a_{rs}f_s(x_1, x_2, \cdots, x_k), \quad r=1, 2, \cdots, k,$$

with rational coefficients a_{rs} such that the system of equations

$$f_s(x_1, x_2, \cdots, x_k)=0, \qquad s=1, 2, \cdots, n,$$

is equivalent to the system

$$g_r(x_1, x_2, \cdots, x_n)=0, \qquad r=1, 2, \cdots, k.$$

The theorem, first stated by Kronecker without proof and first proved by Julius König [Einleitung in die allgemeine Theorie der algebraischen Grössen, Teubner, Leipzig, 1903, p. 234], that a system of arbitrarily many equations in $k-1$ unknowns is always equivalent to a system of at most k equations follows at once as a corollary. The author then discusses the implications of this theorem of Kronecker from the standpoint of ideal theory and closes with the proof of a theorem on the independence of homogeneous polynomials of the same degree. *J. Williamson.*

Lee, H. C. On plane factorizations of pseudo-Euclidean rotations. Quart. J. Math., Oxford Ser. **15**, 7–10 (1944). [MF 10683]

A pseudo-Euclidean rotation is a linear transformation with real coefficients and determinant 1 which leaves the quadratic form $x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_n^2$ invariant. It is shown that such a rotation can be written as a product of $n(n-1)/2$ factors which are either of the form $y_i=x_i\cos\theta-x_j\sin\theta$, $y_j=x_i\sin\theta+x_j\cos\theta$, $y_k=x_k$ for $k\neq i, j$, or of the form $y_i=x_i\cosh\theta+x_j\sinh\theta$, $y_j=x_i\sinh\theta+x_j\cosh\theta$, $y_k=x_k$ for $k\neq i, j$. The number of factors of the second kind can be taken as the smaller of the two numbers p, $n-p$. *R. Brauer* (Toronto, Ont.).

Jacobson, N. Schur's theorems on commutative matrices. Bull. Amer. Math. Soc. **50**, 431–436 (1944). [MF 10612]

Let $N(n)$ denote the number of linearly independent commutative matrices of n rows and columns with elements in a field Φ. In case Φ is algebraically closed, Schur [J. Reine Angew. Math. **130**, 66–76 (1905)] proved that $N(n)=[n^2/4]+1$ and also determined the sets of linearly independent commutative matrices containing $N(n)$ elements. This paper contains a simple derivation of these results and also an extension to the case in which Φ is an arbitrary field. *N. H. McCoy* (Northampton, Mass.).

Abstract Algebra

Hall, M. and Dilworth, R. P. The imbedding problem for modular lattices. Ann. of Math. (2) **45**, 450–456 (1944). [MF 10920]

The authors exhibit a modular lattice which cannot be embedded as a sublattice in any complemented modular lattice. This is a "simple" lattice (that is, lattice whose prime quotients are all mutually projective) of lattice dimension four, which contains a non-Desarguesian plane as an interval sublattice. The proof is based directly on lattice-theoretic computations; it does not involve the introduction of coordinates ("algebra of throws"). It is shown further that any atomic modular lattice is a subdirect union of projective lattices. Finally, a new theorem about harmonic conjugates and their duals in three-dimensional projective geometry is obtained. *G. Birkhoff.*

Birkhoff, Garrett. Subdirect unions in universal algebra. Bull. Amer. Math. Soc. **50**, 764–768 (1944). [MF 11293]

By an algebra A is meant any collection of elements, combined by any number of single-valued operations, each operating on a finite number of elements. The isomorphic representations of A as a subdirect union of other algebras correspond to the sets of congruence relations on A with intersection 0. An algebra is subdirectly irreducible if, in any representation as a subdirect union of a finite or infinite number of algebras H_i, A is isomorphic to at least one H_i. The main result is that any algebra is isomorphic to a subdirect union of subdirectly irreducible algebras. A considerable number of known results are almost immediate consequences of this theorem. For example, any distributive lattice is isomorphic to a ring of sets, any Boolean algebra is isomorphic to a field of sets, any commutative group is a subdirect union of "generalized cyclic" groups (the additive subgroups of the rationals, and those of the rationals mod one). *N. H. McCoy* (Northampton, Mass.).

***Artin, Emil, Nesbitt, Cecil J. and Thrall, Robert M. Rings with Minimum Condition.** University of Michigan Publications in Mathematics, no. 1. University of Michigan Press, Ann Arbor, Mich., 1944. x+123 pp. $1.50.

This is the fourth treatise on the subject to appear in a decade, and the third in English. Having been written "primarily for readers unfamiliar with algebraic methods," it is not as exhaustive as the preceding books by Deuring, Albert, and Jacobson. However, the final chapter includes material not to be found in the earlier books. The point of view taken by the authors stems from the discovery by Artin in 1927 that Wedderburn's theorems could be proved for rings which satisfy the minimal condition on left ideals. Some measure of the progress of this generalization can be gleaned from the fact that, whereas rings expire in favour of algebras at an early point in Deuring, the ring point of view persists nearly to the end of this treatment. The authors strive for and achieve a marked degree of elegance and clarity. The insertion of simple alternative proofs of theorems, when additional hypotheses permit, should make the book particularly valuable to a reader seeking a working knowledge of the subject.

A brief summary by chapters follows. (I) Rings and vector spaces. The fundamental point of view of studying vector spaces over rings is established. (II) Minimum condition. Useful lemmas on the minimum condition and the usual theorems on the radical and idempotents are presented. (III) Matrix representations. Definitions and elementary properties of representations. (IV) Semisimple rings. (V) Simple rings. These two chapters present the Wedderburn theorems. (VI) Kronecker product of spaces and rings. The Kronecker (alias direct or tensor) product is established with novel generality. (VII) Simple rings continued. The deeper theory of simple rings as abridged from a paper of Artin and Whaples [Amer. J. Math. **65**, 87–107 (1943); these Rev. **4**, 129]. (VIII) Splitting fields and crossed products. New proofs are given for the well-known theorems. (IX) Nonsemisimple rings with minimum condition. The final chapter is an introduction to the modern theory of rings with radical, as developed principally by R. Brauer.

[The following correction suggested by the reviewer has been approved by the authors. The definition at the foot of page 53 should be amended as follows: a space is locally finite if any finite set of vectors is in a subspace generated by independent vectors. One confusing misprint was noted. On page 49, third last line, the symbol C_- should be replaced by C_-'.] *I. Kaplansky* (New York, N. Y.).

Sagastume Berra, Alberto E. On the theory of rings. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 174, Serie 2, Revista 11, Vol. 3, num. 1, 107–141 (1944). (Spanish) [MF 11248]

Let $\mathfrak{R}$ be a ring with unit element, R the class of all elements of $\mathfrak{R}$ which are not divisors of zero (on the left or on the right), and U the class of all elements of $\mathfrak{R}$ having either a left or a right inverse, or both. Suppose now that $\mathfrak{R}$ is a ring having the property that $U \subset R$ and that T is a set of elements of $\mathfrak{R}$, closed under multiplication, such that $U \subset T \subset R$. Then a necessary and sufficient condition that there exist a ring of formal quotients with elements of $\mathfrak{R}$ as "numerators" and elements of T as "denominators" is that for each $a \in \mathfrak{R}$, $t \in T$ there exist elements $a_1 \in \mathfrak{R}$, $t_1 \in T$ such that $at_1 = ta_1$. If, as a special case, $\mathfrak{R}$ has no divisors of zero, then $T = \mathfrak{R}$ (with the exception of 0) is a possible choice of T, in which case the ring of quotients is a division ring containing $\mathfrak{R}$. The author has overlooked a discussion by Ore [Ann. of Math. (2) 32, 463–477 (1931)] of this important special case. *N. H. McCoy* (Northampton, Mass.).

Foster, Alfred L. and Bernstein, B. A. Symmetric approach to commutative rings, with duality theorem: Boolean duality as special case. Duke Math. J. 11, 603–616 (1944). [MF 11094]

In any commutative ring R with unity, let $x' = 1 - x$. Then $xx' = 0$, $x + x' = 1$, $(x')' = x$, $(x+y)' = x' + y' - 1$, $(xy)' = x' + y' - x'y'$, $(-x)' = 2 - x'$. It follows that the elements of R form a ring R' isomorphic to R also under the "dual" operations $x \oplus y = x + y - 1$, $x \Delta y = x + y - xy$. The elements of R which are units under both xy and $x \Delta y$ are called double-units; if a is a double-unit, then so are a', $2 - a$, etc. The authors obtain a general duality theorem which reduces, in the case of a Boolean ring (that is, idempotent multiplication), to the usual duality theorem of Boolean algebra. They fail to observe that most of their results and methods apply to commutative nonassociative rings.

G. Birkhoff (Cambridge, Mass.).

Levitzki, Jakob. A characteristic condition for semi-primary rings. Duke Math. J. 11, 367–368 (1944). [MF 10674]

The author's contribution in this paper is to derive the nilpotency of the radical (= union of nilpotent ideals) from a new weak form of the minimal condition, namely, that there are no infinite descending chains of the form $A_1 \supset A_1A_2 \supset A_1A_2A_3 \supset \cdots$, where the A_i are two-sided ideals contained in the radical. [Actually in the author's proof the A_i are nilpotent, so that a still stronger theorem could have been stated.] The best previous result, also given by the author [Duke Math. J. 10, 553–556 (1943); these Rev. 5, 32], assumed the minimal condition for two-sided "semi-nilpotent" ideals. *I. Kaplansky* (New York, N. Y.).

Brauer, Richard. On the arithmetic in a group ring. Proc. Nat. Acad. Sci. U. S. A. 30, 109–114 (1944). [MF 10745]

Let Γ be the group ring over a suitable algebraic number field K of the finite group $\mathfrak{G}$ of order g. The elements of Γ whose coefficients are integers of K constitute a (nonmaximal) order J. The main problem is with the behavior of the extension $(\mathfrak{p})$ in J of a prime ideal $\mathfrak{p}$ of K, and this is equivalent to the study of the modular representations of $\mathfrak{G}$. For every prime ideal divisor $\mathfrak{P}$ of $(\mathfrak{p})$, the residue class ring $J/\mathfrak{P}$ is a complete matric algebra of degree f over a modular field K^*. Thus $\mathfrak{P}$ defines a modular irreducible representation $\mathfrak{F}$ of $\mathfrak{G}$ of degree f, and there is a (1-1)-correspondence between the prime ideal divisors $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$ of $(\mathfrak{p})$ and the irreducible modular representations $\mathfrak{F}_1, \cdots, \mathfrak{F}_k$ of $\mathfrak{G}$. There is a unique representation $\mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \cdots \cap \mathfrak{M}_t$ of $(\mathfrak{p})$ as a direct intersection of ideals which are relatively prime in pairs. The $\mathfrak{P}_i$ dividing $\mathfrak{M}_r$ form a "block" B_r, and the corresponding modular representations $\mathfrak{F}_i$ are said to belong to this block. Every ordinary representation $\mathfrak{Z}_j$ of $\mathfrak{G}$ is associated with exactly one block. Let $\mathfrak{p} \mid p$, where p is a rational prime. If p^a is the highest power of p which divides all degrees f_i of the $\mathfrak{F}_i$ belonging to B_r, then B_r is of type α. If p^α is the highest power of p dividing g, then $d = a - \alpha$ is the defect of B_r. If the group characters of $\mathfrak{G}$ are known, the blocks are easily obtained. Choose one group $\mathfrak{H}_\sigma$ of order p^α from each class of conjugate subgroups of this order of $\mathfrak{G}$, and denote the normalizer of $\mathfrak{H}_\sigma$ by $\mathfrak{N}_\sigma$ ($\sigma = 1, 2, \cdots, l$). For the given prime p, all blocks of $\mathfrak{N}_\sigma$ have at least the defect d. If $\mathfrak{N}_\sigma$ possesses exactly r_σ blocks of defect d, then $\mathfrak{G}$ contains exactly $r_1 + r_2 + \cdots + r_l$ blocks of defect d. There are other results showing that the question of finding all blocks of defect $d > 0$ can be answered by studying groups whose orders are less than g. *C. C. MacDuffee.*

Chevalley, Claude. Some properties of ideals in rings of power series. Trans. Amer. Math. Soc. 55, 68–84 (1944). [MF 9875]

For the purpose of studying the decomposition of a prime ideal of a polynomial ring or a power series ring when the field K of the coefficients is extended to a field Z, the author develops a number of concepts in general field theory that are also of intrinsic interest. An extension Z/K is called separably generated if one of the following equivalent conditions holds. (I) Z/K preserves p-independence. (II) If $y_1, \cdots, y_m$ are elements of Z that are linearly independent over K, then $y_1^p, \cdots, y_m^p$ are linearly independent over K. If Z is a finite extension these conditions are equivalent to the usual condition that Z have a transcendence basis $x_1, \cdots, x_r$ such that Z is separable over $K(x_1, \cdots, x_r)$. Among the results obtained we note the following. A semi-simple algebra over K remains semi-simple if the field K is extended to a separably generated extension Z/K. If $K((X_1, \cdots, X_r))$ denotes the field of power series and $K(X_1, \cdots, X_r)$ that of rational functions over K, then $K((X_1, \cdots, X_r))/K$ and $K((X_1, \cdots, X_r))/K(X_1, \cdots, X_r)$ are separably generated. Now K is called strongly algebraically closed in Z if K is algebraically closed in Z and Z is separably generated. If S/K is a finite algebraic extension of K and K is strongly algebraically closed then S_S is a field. Using these results the author proves that, if $\mathfrak{v}$ is a prime ideal of dimension r in the polynomial ring $K[X_1, \cdots, X_n]$ (the power series ring $K[[X_1, \cdots, X_n]]$) and Z is an extension field of K, then the ideal generated by $\mathfrak{v}$ in $Z[X_1, \cdots, X_n]$ (in $Z[[X_1, \cdots, X_n]]$) is an intersection of primary ideals whose associated prime ideals are all of dimension r. Conditions are obtained that insure that $\mathfrak{v}$ remains prime for every extension Z of K. In the polynomial case the condition is that K be strongly algebraically closed in the quotient field of the integral domain $K[X_1, \cdots, X_n]/\mathfrak{v}$. An analogous condition holds for the power series case. An ideal $\mathfrak{v}$ in $K[[X_1, \cdots, X_n]]$ is said to be definable in a subfield K' of K if there exists a set of generators of $\mathfrak{v}$ consisting of elements of $K'[[X_1, \cdots, X_n]]$. It is proved that for a given ideal $\mathfrak{v}$ there is a least subfield contained in all others in which $\mathfrak{v}$ is definable.

N. Jacobson (Baltimore, Md.).

Hull, Ralph. A theorem on the unit groups of simple algebras. Bull. Amer. Math. Soc. **50**, 405–411 (1944). [MF 10607]

Let k be an algebraic number field of finite degree m; let A (1) be a normal simple algebra of degree n, order n^2, over k, and (2) satisfy the R-condition, that is, $n>2$ or, if $n=2$, at least one infinite prime place of k is unramified in A. For such algebras A the author proves the following theorem. Any two distinct maximal orders of A have distinct groups of units. For $n=2$ the conditions on A are equivalent to: A is not a totally definite quaternion algebra over k. *A. L. Foster* (Berkeley, Calif.).

Tschebotarëw, N. Notice on the theory of algebras. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 405–412 (1942). (English. Georgian summary) [MF 10316]

Simplified proofs are discussed of several known results, mainly concerned with (a) nilpotent and weakly nilpotent algebras, (b) radical and semi-simple algebras, (c) the determinant $D(A)$ of K. Shoda (also F. Ringleb), whose nonvanishing is necessary and sufficient for an algebra A to be simple and normal. In (a) the author considers the theorem: a weakly nilpotent algebra of finite order is nilpotent. A typical theorem discussed in (b) is the following: a semi-simple algebra has a principal unit. *A. L. Foster.*

Beatty, S. and Wales, Muriel. Theory of algebraic functions based on the use of cycles. Trans. Roy. Soc. Canada. Sect. III (3) **38**, 11–30 (1944). [MF 11243]

The authors discuss the field of algebraic functions given by an equation $F(z, u)=0$ over the complex number field C. In modern terminology, the discussion is centered around the splitting of $F(z, u)$ in the z-adic extension of $C(z)$. Thus, without mentioning prime divisors, etc., some simple results on the existence of functions with prescribed zeros are obtained. *O. F. G. Schilling* (Chicago, Ill.).

Kneebone, G. T. On algebraic correspondences. J. London Math. Soc. **18**, 133–137 (1943). [MF 10362]

An algebraic correspondence between parameters x and y is a correspondence established by means of an equation $f(x, y)=0$, where $f(x, y)\in K[x, y]$; K may be any algebraically closed field of characteristic zero. Such a correspondence is said to be (m, n) if some value of x determines n values of y, but none more than n, and if some value of y determines m values of x, but none more than m. It is shown that every such correspondence may be established by an equation

$$f(x, y)\equiv\prod_{i=1}^{r} f_i(x, y)=0,$$

where $f_i(x, y)$ is an irreducible polynomial in $K[x, y]$, the degree of $f(x, y)$ in x is m, and its degree in y is n. Furthermore, any equation which establishes this same correspondence has the form

$$k\prod_{i=1}^{r}\{f_i(x, y)\}^{\lambda_i}=0, \qquad k\in K.$$

The proof is algebraic. *J. L. Dorroh* (Baton Rouge, La.).

Ancochea, Germán. Corps hyperelliptiques abstraits de caractéristique deux. Portugaliae Math. **4**, 119–128 (1943). [MF 11266]

An explicit determination of invariants and groups of automorphisms for hyperelliptic fields of genus 2 over algebraically closed fields of characteristic 2. *C. Chevalley.*

Jacobson, N. An extension of Galois theory to nonnormal and non-separable fields. Amer. J. Math. **66**, 1–29 (1944). [MF 9936]

M. Krasner [Sur la théorie de la ramification des ideaux des corps non galoisiens de nombres algébriques, Thesis, Paris, 1938] and L. Kaloujnine [C. R. Acad. Sci. Paris **214**, 597–599 (1942); these Rev. **4**, 130] both have given generalizations of Galois theory to the case of nonnormal separable extensions. The present paper gives a generalization which has the decisive advantage of including the case of nonseparable extensions.

Previous investigations of the author have shown that, for purely unseparable extensions K/k generated by adjunctions of pth roots of elements of the basic field (where p is the characteristic), the derivations of K/k can be made to play a role similar to the role played by the automorphisms of K/k in the classical theory. The starting point of the present paper is the remark that both the notions of automorphism and of derivation of a field P can be included as special cases in the notion of representation of the field P in itself, or self-representation of P. The author is therefore led to undertake a systematic study of the self-representations of a field.

Most of the results are formulated not in terms of representations but in terms of an equivalent notion, that of a composite of P with itself. By a composite of P with itself is meant a system formed by a commutative ring K and two isomorphisms S and T of P with subfields of K which satisfy the following conditions: (1) K is generated by P^S and P^T; (2) $1^S=1^T$; (3) K is of finite rank over P^T. Such a composite is called nonsingular if K has the same rank over P^S and P^T. To the Kronecker multiplication of self-representations there corresponds a multiplication of composites. A composite (K, S, T) is said to cover a composite (K', S', T') if there exists a homomorphism $\bar{\omega}$ of K onto K' such that $S'=\bar{\omega}S$ and $T'=\bar{\omega}T$. A composite is called closed, or a Galois composite, if it covers its product by itself. An element α of P is called a fixed element for the composite (K, S, T) if it is represented by a diagonal matrix whose diagonal coefficients are equal to α in the self-representation which corresponds to the composite; an equivalent condition is that $\alpha^S=\alpha^T$. The fixed elements of a Galois composite Γ form a subfield ϕ_Γ of P over which P is of finite rank. The first main theorem of the paper asserts that the correspondence $\Gamma\to\phi_\Gamma$ is a one-to-one correspondence between the Galois composites of P and the subfields ϕ_Γ over which P is of finite rank. Furthermore, $\phi_\Gamma\subset\phi_{\Gamma'}$ if and only if Γ covers Γ'.

A composite (K, S, T) is said to be semi-simple if K is a direct sum of fields, simple if K is a field. A necessary and sufficient condition for a Galois composite Γ to be semi-simple is for P to be separable over ϕ_Γ. This being the case, the fields in the direct sum into which K can be decomposed give rise to a system of simple composites Γ_i of P, and the products $\Gamma_i\times\Gamma_j$ can be represented as sums of Γ_k's. This leads to a law of composition of hypergroups in the set of the Γ_i's; this hypergroup is isomorphic to the hypergroups introduced by Krasner and Kaloujnine.

A composite (K, S, T) is called one-dimensional if $K=P^T$. A necessary and sufficient condition for P to be normal over a subfield ϕ over which it is finite is that the simple composites whose fields of fixed elements contain ϕ be all one-dimensional. By combining his criteria for separable and for normal extensions, the author obtains the classical Galois theory as a particular case of his results. *C. Chevalley.*

Ore, Oystein. Galois connexions. Trans. Amer. Math. Soc. **55**, 493–513 (1944). [MF 10469]

The author defines abstractly a Galois connexion between partially ordered sets P and Q as a pair of transformations ϕ and θ from P to Q and vice versa, such that (i) $p \geqq p'$ implies $\phi(p) \leqq \phi(p')$ while $q \geqq q'$ implies $\theta(q) \leqq \theta(q')$ and (ii) $\theta(\phi(p)) \geqq p$ and $\phi(\theta(q)) \geqq q$ for all $p \in P$, $q \in Q$. He points out that the operations $p \to \theta(\phi(p))$ and $q \to \phi(\theta(p))$ are always closure operations in the sense of E. H. Moore; if $p = \theta(\phi(p))$ and $q = \phi(\theta(q))$ for all p, q, the connexion is called perfect; if $P = Q$ and $\phi = \theta$, it is called self-dual. The reviewer has shown [Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940, §32; these Rev. **1**, 325] that every binary relation between elements of sets I and I^* yields a Galois connexion between the Boolean algebras B and B^* of all their subsets, and that a symmetric relation on I gives a self-dual connexion between B and itself. The author proves, conversely, that all Galois connexions between I and I^* and all self-dual Galois connexions of B arise in this way. Similarly, he proves that any complete lattice with a "polarity" can be generated by a symmetric anti-reflexive relation between elements. Other theorems describe necessary and sufficient conditions for the closure relation defined by a Galois connexion to have certain properties. Numerous interesting illustrations are given.

G. Birkhoff (Cambridge, Mass.).

Everett, C. J. Closure operators and Galois theory in lattices. Trans. Amer. Math. Soc. **55**, 514–525 (1944). [MF 10466]

The author discusses Galois connexions between sets [in the sense of the preceding review] and closure operators $S \to \bar{S}$ on sets. He shows first that every closure is a Galois closure and then establishes the equivalence between Galois connexions of sets and binary relations on elements described in the preceding review. Two extension theorems on closure operators are also given. It is shown that a subgroup of the Galois group of an infinite normal extension of a field F is closed in the Krull topology if and only if it is "closed" under the relation $\phi(a) = a$. He corrects an analogous result of the reviewer regarding closed subspaces of a Banach space and its conjugate space and proves a new but similar result on primary Abelian groups. He unifies these results by giving a necessary and sufficient condition that Galois closure on a family of subsets S of a set Ω should be equivalent to topological closure: it is that $S \leqq \bar{S}_1 \cup \cdots \cup \bar{S}_n$ implies $\bar{S} \leqq \bar{S}_1 \cup \cdots \cup \bar{S}_n$. *G. Birkhoff.*

NUMBER THEORY

Whitlock, W. P., Jr. Rational right triangles with equal areas. Scripta Math. **9**, 265–267 (1943). [MF 11036]

Continuation of the article in the same journal **9**, 155–161 (1943); these Rev. **5**, 199.

Kulakoff, A. A. Sur les nombres de la forme $a^m \pm b^m$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 43–45 (1943). [MF 11172]

This is a continuation of previous work by the author [Enseignement Math. **37**, 73–76 (1938)]. A typical result is the following. Let a, b, k be positive integers, relatively prime in pairs, with $ab \neq 1$. Then, if integers m exist such that k divides $a^m + b^m$, the least such m divides all others.

I. Niven (Lafayette, Ind.).

Radhakrishna Rao, C. Extension of the difference theorems of Singer and Bose. Science and Culture **10**, 57 (1944). [MF 11119]

The author constructs a set of $k = (m^t - 1)/(m - 1)$ (m a prime power) integers such that the $k(k-1)$ differences arising from this set contain every residue modulo

$$(m^{t+1} - 1)/(m - 1)$$

(misprinted in the paper) except 0 exactly

$$\lambda = (m^{t-1} - 1)/(m - 1)$$

times. Similarly, he constructs a set of m^{t-1} integers such that the differences arising from this set contain every residue class modulo $m^t - 1$ which is not divisible by $(m^t - 1)/(m - 1)$ exactly m^{t-2} times. These results are generalizations of results obtained by Singer [Trans. Amer. Math. Soc. **43**, 377–385 (1938)] and R. C. Bose [Indian J. Math. **2**, 15 (1943)], respectively. The author announces his results without detailed proofs, which will be published elsewhere. *H. B. Mann* (Columbus, Ohio).

Pillai, S. S. Highly abundant numbers. Bull. Calcutta Math. Soc. **35**, 141–156 (1943). [MF 11012]

The author calls a number N highly abundant of the rth order if $\sigma_{-r}(N) > \sigma_{-r}(N')$ for all $N' < N$, where $\sigma_s(N)$ is the sum of the sth powers of the divisors of N. A highly abundant number of order zero is a highly composite number of Ramanujan. Properties of the exponents in the decomposition of N into a product of prime powers are obtained and there is some discussion of the distribution of highly abundant numbers. The inequality used to obtain formula (9) appears to be incorrect. *H. S. Zuckerman.*

Pillai, S. S. On the smallest primitive root of a prime. J. Indian Math. Soc. (N.S.) **8**, 14–17 (1944). [MF 11272]

The author proves that for infinitely many primes p the least primitive root of p is greater than $c \log \log p$.

P. Erdős (Princeton, N. J.).

Sathe, L. G. On a congruence property of the divisor function. I. J. Indian Math. Soc. (N.S.) **7**, 143–145 (1943). [MF 10929]

Let $N(k, x)$ denote the number of $n \leqq x$ for which $d(n)$, the number of divisors of n, is a multiple of k. It is proved that, for any odd prime k,

$$N(k, x) = A_k x + O(x^{1/(k-1)} \log x),$$

where

$$A_k = 1 - \prod_p \{1 - (p-1)/(p^k - 1)\},$$

p running through all primes. The author states that this corrects an erroneous result by S. S. Pillai [J. Indian Math. Soc. (N.S.) **6**, 118–119 (1942); these Rev. **4**, 210]. There is a misprint in the equation for A_k in the statement of the theorem on page 143. *I. Niven* (Lafayette, Ind.).

Sathe, L. G. On a congruence property of the divisor function. II. J. Indian Math. Soc. (N.S.) **7**, 146–151 (1943). [MF 10930]

Extending the result stated in the preceding review, the author proves: (1) $N(k, x) \neq o(x)$ for any positive integer k; (2) $N(2k, x) \sim N(k, x)$ for k odd; (3) for $k = m_1 m_2$, $m_1 < m_2$, m_1 and m_2 being odd primes,

$$N(k, x) = C_k x + O(x^{1/(m_1-1)+\epsilon})$$

for every $\epsilon > 0$, C_k being a positive constant.

I. Niven (Lafayette, Ind.).

Ramanathan, K. G. Congruence properties of Ramanujan's function $\tau(n)$. Proc. Indian Acad. Sci., Sect. A. **19**, 146–148 (1944). [MF 10981]

Ramanathan, K. G. Congruence properties of $\sigma(n)$, the sum of the divisors of n. Math. Student **11**, 33–35 (1943). [MF 10997]

These papers contain simple proofs of the congruences

$$\sigma(kn-1) \equiv \tau(kn-1) \equiv 0 \pmod{k},$$

where k is any divisor of 24 greater than 2. These are based on the congruence theory of power series and on well-known identities of the generating functions of σ and τ.

D. H. Lehmer (Berkeley, Calif.).

Levi, B. Some elementary notions of the theory of numbers. Math. Notae **4**, 65–79 (1944). (Spanish) [MF 11023]

An expository article on Gaussian complex integers and some related matters, written to prepare the reader for the paper reviewed below. *H. W. Brinkmann.*

Uspensky, J. V. A new proof of Jacobi's theorem. Math. Notae **4**, 80–89 (1944). (Spanish) [MF 11024]

It is known that the theorem of Jacobi on the number of representations of a positive integer n as a sum of four squares can be deduced from Eisenstein's formula for the number of primitive representations as the sum of four squares. It is Eisenstein's formula for an odd n that is derived in this paper. To do this, the author shows that all primitive solutions of the equation (1) $t^2 = 4x^2 + 4y^2 + z^2$ with t odd are obtained by the formulas $x + yi = \alpha\bar{\beta}$, $t = \alpha\bar{\alpha} + \beta\bar{\beta}$, $z = \beta\bar{\beta} - \alpha\bar{\alpha}$, where α, β are relatively prime Gaussian complex integers; each solution of (1) occurs exactly four times. If we set $\alpha = a + bi$, $\beta = c + di$, we get $t = a^2 + b^2 + c^2 + d^2$, where a, b, c, d are subject to the restriction that α, β are relatively prime. This restriction is removed by a simple device (although a, b, c, d continue to be relatively prime) and thus a relation is established between the number of primitive representations of t as a sum of four squares and the number of solutions of (1). It remains only to derive a formula for the number of solutions of (1); the author does this by using a theorem of Liouville which itself can be derived either by using an identity in the theory of elliptic functions or by the well-known elementary methods discovered by Liouville.

H. W. Brinkmann (Swarthmore, Pa.).

Wormser, Guy. Sur les nombres premiers représentables par des polynomes du second degré. C. R. Acad. Sci. Paris **217**, 241–242 (1943). [MF 11108]

Write $f_{n,p}(x, y) = n(x^2 + xy) + py^2$ with $n = 1$, 2 or 3 and p a positive integer. Let $z_{n,p}$ and $Z_{n,p}$ be the positive zeroes, respectively, of the polynomials $x^2 - f_{n,p}(x, 1)$ and $f_{n,p}(x, 1) - p^2$, while $M_{n,p} = \min(2Z_{n,p}, p)$. The author proves the theorem: if x and y are two integers such that $f_{n,p}(x, 1)$ is a prime for all x with $0 \leqq x \leqq z_{n,p}$, if y is prime to nx, and the point (x, y) is interior to the ellipse $f_{n,p}(x, y) = M_{n,p}^2$, then $f_{n,p}(x, y)$ is a prime number. *B. W. Jones.*

Segre, B. A parametric solution of the indeterminate cubic equation $z^3 = f(x, y)$. J. London Math. Soc. **18**, 226–233 (1943). [MF 10877]

The author proves the following theorem. Let $f(x, y)$ be any given cubic polynomial in x, y with rational coefficients, which is not expressible as a polynomial in a single linear function of x and y. Then the equation $z^3 = f(x, y)$ has an infinity of rational solutions in x, y, z, given rationally in terms of one or more parameters. From this theorem these results follow. (1) If a rational solution of a given indeterminate cubic equation $F(x, y, z) = 0$ is known, which does not annul all the first and second partial derivatives of $F(x, y, z)$, so that $F = 0$ is not a curve for any choice of coordinates, then it is possible to obtain an infinity of rational solutions, expressed in terms of one or more parameters. (2) An indeterminate ternary cubic equation which cannot be reduced to an equation involving less than three variables has either no rational solution or an infinity of rational solutions. Both cases are in fact possible.

B. W. Jones (Ithaca, N. Y.).

Mordell, L. J. A rational parametric solution of $z^2 - k = ax^3 + bx^2y + cxy^2 + dy^3$. J. London Math. Soc. **18**, 222–226 (1943). [MF 10876]

The author finds a parametric solution of the equation

$$z^2 - k(1 + px)^2 = ax^3 + bx^2y + cxy^2 + dy^3,$$

whose existence for $k = 1$ has been proved by Segre. A short resume of recent results is also given. *B. W. Jones.*

Mordell, L. J. The minimum of a binary cubic form. I. J. London Math. Soc. **18**, 201–210 (1943). [MF 10873]

In a recent paper [Proc. London Math. Soc. (2) **48**, 198–228 (1943); these Rev. **5**, 172], the author proved the following theorem. Let $x = \alpha\xi + \beta\eta$, $y = \gamma\xi + \delta\eta$, where ξ, η assume all integral values, and $\alpha, \beta, \gamma, \delta$ are real numbers such that $\alpha\delta - \beta\gamma = 1$, be any lattice L of determinant unity. Then a point P of L other than the origin O lies in the region R defined by $|x^3 - xy^2 - y^3| \leqq 1$; the equality sign is required for, and only for, the two lattices $x = \xi$, $y = \eta$ and $(3\theta^2 - 1)x = -\xi - (\theta + 3)\eta$, $(3\theta^2 - 1)y = -3\theta\xi + \eta$, where θ is the real root of $t^3 = t + 1$. Now it is proved that P may be chosen in the finite part of R cut off by the parts of the lines $y = 1$, $3\theta^2 x - y = 3\theta - 1$ lying in the first quadrant, and their images in O. The proof uses the same geometric ideas, but it is simpler and involves less calculation than the former one. The author remarks that a further simplification is obtained by taking R to be the region $|x^3 + y^3| \leqq 1$ instead of $|x^3 - xy^2 - y^3| \leqq 1$; this will be the subject of a further paper [cf. the two following reviews]. *C. L. Siegel.*

Mordell, L. J. The minimum of a binary cubic form. II. J. London Math. Soc. **18**, 210–217 (1943). [MF 10874]

[Cf. the preceding and the following reviews.] A new proof of Minkowski's theorem on the product of two non-homogeneous linear forms $f(x, y)$, $g(x, y)$, with determinant 1, is given by applying Minkowski's theorem on convex bodies to the three-dimensional region

$$\lambda|zf(x/z, y/z)| + \lambda^{-1}|zg(x/z, y/z)| \leqq 1, \qquad |z| \leqq 2,$$

λ any positive constant. *C. L. Siegel* (Princeton, N. J.).

Mordell, L. J. The product of two non-homogeneous linear forms. III. J. London Math. Soc. **18**, 218–221 (1943). [MF 10875]

[Cf. the two preceding reviews.] A simplified proof and a refinement of the following theorem: a point of L other than the origin lies in the region $|x^3-x^2y-2xy^2+y^3|\leqq 1$; the equality sign is required for, and only for, the two lattices $x=\xi$, $y=\eta$ and $(\theta-\phi)x=\theta\phi^2\xi+\psi\eta$, $(\theta-\phi)y=-\theta^2\xi-\theta\phi^2\eta$, where θ, ϕ, ψ are the three roots of $t^3+t^2=2t+1$ and $\theta>\phi>\psi$ [cf. the preceding review]. *C. L. Siegel.*

Siegel, Carl Ludwig. On the theory of indefinite quadratic forms. Ann. of Math. (2) **45**, 577–622 (1944). [MF 10924]

Let $\mathfrak{S}$ be a nonsingular m-rowed symmetric matrix with integer elements and $\Gamma(\mathfrak{S})$ the group of all integral matrices $\mathfrak{U}$ satisfying the equation $\mathfrak{S}[\mathfrak{U}]=\mathfrak{S}$ (that is, $\mathfrak{U}'\mathfrak{S}\mathfrak{U}=\mathfrak{S}$). If $\mathfrak{X}$ and $\mathfrak{W}$ are m by n real nonsingular matrices and $\mathfrak{W}$ is symmetric, $\mathfrak{S}[\mathfrak{X}]=\mathfrak{W}$ defines an $\frac{1}{2}m(m-1)$ dimensional surface and $\rho(\mathfrak{S})$ is the volume of a fundamental domain on the surface $\mathfrak{S}[\mathfrak{X}]=\mathfrak{W}$ with respect to $\Gamma(\mathfrak{S})$ computed with a certain given volume element. Let $\mathfrak{G}$ be a given real matrix with m rows and n columns and of rank n, while $\Gamma(\mathfrak{S},\mathfrak{G})$ is the subgroup of $\Gamma(\mathfrak{S})$ defined by the condition $\mathfrak{U}\mathfrak{G}=\mathfrak{G}$. Moreover, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are said to be associate relative to $\Gamma(\mathfrak{S})$ if for some element $\mathfrak{U}$ of $\Gamma(\mathfrak{S})$ it is true that $\mathfrak{G}_2=\mathfrak{U}\mathfrak{G}_1$.

The author defines the "measure of the representations of $\mathfrak{T}$ by $\mathfrak{S}$" as

$$\mu(\mathfrak{S},\mathfrak{T})=\sum_{\mathfrak{G}}(\mathfrak{S},\mathfrak{G})/\rho(\mathfrak{S}),$$

where $\mathfrak{G}$ runs over a complete set of nonassociate integral solutions of $\mathfrak{S}[\mathfrak{G}]=\mathfrak{T}$ of rank n. Let $A_q(\mathfrak{S},\mathfrak{T})$ be the number of modulo q incongruent solutions $\mathfrak{G}$ of the congruence $\mathfrak{S}[\mathfrak{G}]=\mathfrak{T}$ (mod q), where q is an arbitrary positive integer, and

$$\alpha_p(\mathfrak{S},\mathfrak{T})=\lim_{l\to\infty}q^{\frac{1}{2}n(n+1)-ln}A_{q^l}(\mathfrak{S},\mathfrak{T})$$

for $q=p^l$, p being a prime. The author's principal result is the following theorem. Let r, $m-r$ be the signature of $\mathfrak{S}$ and $n\leqq r$, $n\leqq m-r$, $2n+2<m$. Then

$$\mu(\mathfrak{S},\mathfrak{T})=\prod_{p}\alpha_p(\mathfrak{S},\mathfrak{T}),$$

where p runs over all primes. A corresponding result is proved for primitive solutions. This is a refinement of a result in a former paper [Ann. of Math. (2) **36**, 527–606 (1935); also (2) **37**, 230–263 (1936)], and consists chiefly in showing that all quadratic forms in the same genus have the same representation measures when the conditions of the theorem hold, whereas the former results were concerned with a sum over the classes of the genus. *B. W. Jones.*

Benneton, Gaston. Une arithmétique des biquaternions. C. R. Acad. Sci. Paris **216**, 262–264 (1943). [MF 10959]

Let X, Y denote Hamiltonian quaternions. The author studies biquaternions $A=(X, Y)$, with the multiplication law $(X, Y)\cdot(Z, W)=(XZ-\bar{W}Y, WX+Y\bar{Z})$. The conjugate of A is $(\bar{X}, -Y)$, and the norm $N(A)$ of A is the sum of the squares of its eight components; $N(AB)=N(A)\cdot N(B)$. Many results are stated on the factorization of integral biquaternions A, proofs no doubt to appear later. In particular, the biquaternions $(1, Y)$ or $(0, Y)$ seem to play the role of pure quaternions; the number of biquaternions of an odd prime norm p is 16 times the number of solutions of $x^2+y^2+z^2+t^2+1\equiv 0$, $x^2+y^2+1\equiv 0$, $x^2+1\equiv 0 \bmod p$. The number of primitive representations of an integer as a sum of 8 squares, with certain parity restrictions, can be deduced. *G. Pall* (Montreal, Que.).

Rédei, L. Zur Frage des Euklidischen Algorithmus in quadratischen Zahlkörpern. Math. Ann. **118**, 588–608 (1942). [MF 10716]

The problem of the existence of the Euclidean algorithm in quadratic fields $\mathfrak{P}(\sqrt{m})$ is still unsolved only in the case that $m=p$ is a prime of form $4n+1$. For $p=24n+5$ the problem was solved by Behrbohm and Rédei [J. Reine Angew. Math. **174**, 192–205 (1936)] and for $p=24n+13$ by A. Brauer [Amer. J. Math. **62**, 697–716 (1940); these Rev. **2**, 146] with the exception of the cases $p=61$ and $p=109$. Using a new method the author proves the nonexistence of the algorithm in the cases $p=61$ and 109, and for the primes of form $24n+17$ if $p>41$. Thus only the case $p=24n+1$ remains unsolved. On the other hand, the author proves that the algorithm does exist if $p=73$. He is able to prove also that $\mathfrak{P}(\sqrt{97})$ is a field with Euclidean algorithm. *A. Brauer* (Chapel Hill, N. C.).

Gut, Max. Zur Theorie der Klassenkörper der Kreiskörper, insbesondere der Strahlklassenkörper der quadratisch imaginären Zahlkörper. Comment. Math. Helv. **15**, 81–119 (1943).

Let k be a field of algebraic numbers, and let $\mathfrak{F}$ be an ideal in k. Denote by $K\{\mathfrak{F}\}$ the class field over k determined by the congruence group of ideals modulo $\mathfrak{F}$ and by $k\{\mathfrak{F}\}$ the largest absolutely Abelian subfield of $K\{\mathfrak{F}\}$. The present paper is divided in two parts: (a) study of the field $k\{1\}$ when k is a cyclotomic field, in particular, when k is quadratic or composed of quadratic fields and when k is absolutely cyclic of odd degree or composed of such fields; (b) study of $k\{\mathfrak{F}\}$ for arbitrary $\mathfrak{F}$ when k is imaginary quadratic.

(a) Let $c(m)$ be the field of mth roots of unity, and let $m=l_1^{a_1}\cdots l_r^{a_r}$ be the decomposition of m in prime factors. A subfield k of $c(m)$ is called an "Ausgangs-Kreiskörper" when it is free composite of subfields of $c(l_1^{a_1}), \cdots, c(l_r^{a_r})$. If $k\subset c(m)$, it is proved that $k\{1\}$ is the smallest "Ausgangs-Kreiskörper" to contain k. This theorem is applied to determine $k\{1\}$ explicitly in the cases mentioned above. (b) Let k be an imaginary quadratic field and let $\mathfrak{F}$ be an ideal in k. The ideal $\mathfrak{F}$ is called a "Stammideal" if its norm is a power of a prime. It is clear that $\mathfrak{F}$ can be represented in the form $\mathfrak{F}=\mathfrak{F}_1\cdots\mathfrak{F}_t$, where each $\mathfrak{F}_i$ is a Stammideal and where the norms of the $\mathfrak{F}_i$'s are pairwise relatively prime. Under these conditions, the author proves that $k\{\mathfrak{F}\}$ is the composition of $k\{\mathfrak{F}_1\}, \cdots, k\{\mathfrak{F}_t\}$ [observe that it is not true that $K\{\mathfrak{F}\}$ is the composition of $K\{\mathfrak{F}_1\}, \cdots, K\{\mathfrak{F}_t\}$]. The author determines explicitly $k\{\mathfrak{F}\}$ when $\mathfrak{F}$ is a Stammideal and gives at the same time some indications on the structure of $K\{\mathfrak{F}\}$.

When k is an imaginary quadratic field, the field $K\{\mathfrak{F}\}$ can be generated over k by adjunction of roots of equations which in many cases can be constructed explicitly from the theory of elliptic functions. The author applies his considerations to the problem of finding out how these equations can be reduced by adjoining roots of unity to the basic field. *C. Chevalley* (Princeton, N. J.).

Gut, Max. Zur Theorie der Strahlklassenkörper der quadratisch reellen Zahlkörper. Comment. Math. Helv. **16**, 37–59 (1944).

We use the same notation as in the paper reviewed above. The present paper is concerned with the explicit determination of $k\{\mathfrak{F}\}$ when k is a real quadratic field. ($\mathfrak{F}$ is here meant to include the two infinite prime spots of k.) The theorem which, in the case of an imaginary quadratic k, allowed the reduction of the general case to the case where $\mathfrak{F}$ is a Stammideal still holds in the present case. The determination of $k\{\mathfrak{F}\}$ is then achieved explicitly in the case where $\mathfrak{F}$ is a Stammideal.

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two ideals whose norms are relatively prime. An explicit formula is given for the degree of $K\{\mathfrak{F}_1\mathfrak{F}_2\}$ over the compositum Z of $K\{\mathfrak{F}_1\}$ and $K\{\mathfrak{F}_2\}$. It is proved that the Galois group of $K\{\mathfrak{F}_1\mathfrak{F}_2\}/Z$ cannot be reduced by adjunction of roots of unity to Z. In contrast to this last result, it is also shown that, $\mathfrak{P}$ being some fixed prime ideal, there exists an exponent w_0 such that, for any $w \geqq w_0$, $K\{\mathfrak{P}^w\}$ can be obtained from $K\{\mathfrak{P}^{w_0}\}$ by adjunction of roots of unity. *C. Chevalley* (Princeton, N. J.).

Siegel, Carl Ludwig. Algebraic integers whose conjugates lie in the unit circle. Duke Math. J. **11**, 597–602 (1944). [MF 11093]

Let θ be an algebraic integer ($\theta \neq 0, \theta \neq 1$) having the property that all its conjugates lie inside the unit circle; θ is real and can be assumed to be positive. Then $\theta > 1$. It has been proved that the set of all numbers θ is closed [R. Salem, Duke Math. J. **11**, 103–108 (1944); these Rev. **5**, 254]. Therefore there exists in the set a smallest number $\theta > 1$. The author proves that the two smallest numbers of the set are the positive zero θ_1 of $x^3 - x - 1$, and the positive zero θ_2 of $x^4 - x^3 - 1$; both θ_1 and θ_2 are isolated points of the set. He also proves that any other number of the set is larger than $\sqrt{2}$. *R. Salem* (Cambridge, Mass.).

Linnik, U. V. The zeros of L-series, power non-residues and the class number of $k(\sqrt{-D})$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 123–124 (1943). [MF 10041]

A statement is given of results obtained by using the "analogy property" of L-series [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 107–109 (1943); these Rev. **5**, 142]. Let $\chi(n)$ be a nonprincipal character mod D, N_0 the least positive integer for which $\chi(n) \neq 1$; if $N > D^{\varepsilon}$, then the number of zeros of $L(s)$ in the square $1 - (\ln D)^{-a} \leqq \sigma \leqq 1$, $|t| \leqq \frac{1}{2}(\ln D)^{-a}$ exceeds $c_3 r \cdot \ln D/(\ln \ln D)^2$. A connection is stated between the least cubic nonresidue mod p and the magnitude of the class number $h(-p)$. *G. Pall* (Montreal, Ont.).

Popov, A. I. Several series containing primes and roots of $\zeta(s)$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 362–363 (1943). [MF 11074]

Results are announced of formulas involving certain series connected with the roots of $\zeta(s)$. A typical result is

$$\sum_{n>x} \Lambda(n)\frac{\{n/x\} - \{n/x\}^2}{n^2} = \frac{2 - \lg 2\pi}{x} + \sum_{\rho} \frac{x^{\rho-2}}{\rho(\rho-1)} + \sum \frac{k+1-2k\zeta(2k+1)}{2k(k+1)(2k+1)x^{2k+2}},$$

where $\Lambda(n)$ has the customary meaning in analytic number theory; $\{t\}$ is the fractional part of t; the first summation on the right side extends over all complex roots ρ of $\zeta(s)$; the series on the left extends over all integers n greater than the continuous variable x, $x \geqq 1$. It is stated that other formulas can be derived from these determining the numerical values of certain series whose terms are functions of primes, and that the proofs are based on the theory of Dirichlet series. *I. Niven* (Lafayette, Ind.).

Banerjee, D. P. On some formulae in analytical theory of numbers. Bull. Calcutta Math. Soc. **36**, 49–50 (1944). [MF 11217]

Typical formulas are

$$\zeta^2(s) \prod_p \{1 - (1-k)p^{-s}\} = \sum_{n=1}^{\infty} (d(n^k)/n^s)$$

and

$$d(n^k) = \sum_{uv=n} d(u)\mu(v)(1-k)^{\nu(v)},$$

where k is taken to be a proper fraction (presumably it might be an arbitrary complex number), $\nu(v)$ is the number of prime factors of v, and $d(n^k)$ is defined by $d(n^k) = \Pi(1+ke)$, $n = \Pi p^e$. *L. Carlitz* (Durham, N. C.).

Roussel, André. Sur les applications arithmétiques de certains théorèmes dus à É. Picard. C. R. Acad. Sci. Paris **216**, 227–229 (1943). [MF 10956]

The title refers to results of É. Picard expressing in terms of integrals the number of solutions of $f(x, y) = g(x, y) = 0$ contained in a given domain. It is indicated that certain number-theoretic propositions can be stated in terms of the number of solutions of such a pair of equations. The author suggests that Fermat's last theorem can be treated thus, but admits the difficulty of studying the resulting integrals. *I. Niven* (Lafayette, Ind.).

THEORY OF GROUPS

***Wigner, Eugen. Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren.** J. W. Edwards, Ann Arbor, Michigan, 1944. viii+332 pp. $6.45.

This is a reprint of the original which appeared as vol. 85 of the collection Die Wissenschaft, Sammlung von Einzeldarstellungen aus den Gebieten der Naturwissenschaft und der Technik; publisher: F. Vieweg, Braunschweig, 1931.

Miller, G. A. Groups containing less than twenty-eight non-invariant operators. Proc. Nat. Acad. Sci. U. S. A. **30**, 275–279 (1944). [MF 11042]

Most difficulty is caused by the enumeration of groups containing twenty-four noninvariant operators, of which there are seventeen. The enumerations for twenty-five, twenty-six and twenty-seven are almost immediate. *G. de B. Robinson* (Ottawa, Ont.).

Miller, G. A. Possible numbers of non-invariant operators of a group. Proc. Nat. Acad. Sci. U. S. A. **30**, 114–117 (1944). [MF 10746]

The author pursues his study of groups as they are determined by the number of their noninvariant operators. In particular, the alternating group of order 60 is the only group having exactly 59 such operators. *G. de B. Robinson* (Ottawa, Ont.).

Fite, William Benjamin. The degree of a linear homogeneous group. Trans. Amer. Math. Soc. **56**, 1–6 (1944). [MF 10788]

Let a linear homogeneous group G be of order g and of degree n, let its central H be of order h, and let its central quotient group G' have an invariant operation of highest order b, whose powers correspond to the transforming of a substitution T of G into its b distinct conjugates $A^{-x}TA^{x}$, $x=1, 2, \cdots, b$. The number b is the index of a group G_1 in which T is invariant; it is the order of the invariant commutator $T_1=T^{-1}A^{-1}TA$, and is therefore divisible by the highest order c of any multiplier of this invariant commutator. (We have introduced the letters b and c to avoid the confusion which arises from the author's notation whereby β and t^{β} are used interchangeably for the same thing in different theorems.) For an irreducible group it is known that $n|g$ [Frobenius], that $nh|g$ [Schur] and that $b|n$ [Fite]. The author shows the following. (1) For any group G we have $c\leqq n$, and, if $c=n$, then G is irreducible. (2) For a p-group of odd order, $b|n$, even if the commutator T_1 is not assumed to be invariant, but if, instead, T_1 is in G_1 and $T_1^{-1}A^{-1}T_1A$ is invariant. (3) In an irreducible group G there are b distinct multipliers of T corresponding to the systems of imprimitivity of G; hence the subgroup G_1 is Abelian if and only if $b=n$. (4) If $b=n$ and G is irreducible, then G can be written in monomial form. (5) In further theorems some relations between the numbers g and n and the structure of the subgroups H and K are then stated for odd p-groups with cyclic commutator group K. *J. S. Frame.*

Kulakoff, A. A. Sur un critérium de non-simplicité d'un groupe fini. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 3–4 (1943). [MF 11181]

If an abstract group G of order g contains a proper subgroup G' (of order $g'<g$) having more than one element R_C in its central C, and if $\sum(g/h_S)<(2r-1)g'$, then G is composite. Here h_S is the number of conjugates of S in G, r is the number of classes of G, and the summation extends over all elements S of G'. The proof depends on evaluating in two different orders the sum

$$\sum\sum\chi_S^{u}\chi_S^{u}{}^{-1}=\sum(g/h_S)$$

over all S in G' and over the characters χ_S^{u} of all the irreducible representations Γ_u of G. For fixed u, the sum with respect to S in G' is the sum of the squares of the multiplicities of the irreducible representations of G' which occur in the reduction of the corresponding subgroup $\Gamma_{u,G'}$ of Γ_u. If the assumed inequality holds, then at least one representation $\Gamma_{u'}$ of G, besides the identity Γ_1, must remain irreducible for G'. In this representation the matrix of the assumed element R_C of the central of G' must be a multiple of the unit matrix. The existence of a normal subgroup of G is readily proved, whether the degree of $\Gamma_{u'}$ be 1 or more.

J. S. Frame (East Lansing, Mich.).

Baer, Reinhold. Crossed isomorphisms. Amer. J. Math. **66**, 341–404 (1944). [MF 10909]

The author assumes that with every element x of a group A there is associated an endomorphism $e(y)$ of A which maps x on $x^{e(y)}$, x in A. A crossed isomorphism or an $e(x)$-isomorphism γ of A on a group B is a (1-1)-mapping $z\to z^{\gamma}$ of A on B such that $x^{\gamma}y^{\gamma}=(x^{e(y)}y)^{\gamma}$ for any two elements x, y of A. The $e(y)$ themselves have to satisfy certain conditions in order that a crossed isomorphism with the given system $e(y)$ may exist. In particular, every $e(y)$ has to be a proper automorphism of A. An automorphism is called an integral automorphism if every subgroup S is mapped on itself. Of course, this is a very strong restriction. It implies that every element is mapped on a power of itself. If every $e(x)$ is an integral automorphism of A, then the $e(x)$-automorphism γ of A on B is called an integral crossed isomorphism. If the additional assumption is made that γ maps every element of A upon an element of equal order in B, then γ is a coset preserving crossed isomorphism. By this we mean that a subgroup S of A is a subgroup if and only if S^{γ} is a subgroup of B, and two elements u, v of A lie in the same coset modulo S if and only if u^{γ} and v^{γ} lie in the same coset modulo S^{γ}. The author studies crossed isomorphisms in much detail, in the main restricting himself to integral crossed isomorphisms and coset preserving crossed isomorphisms. We can indicate here only the general trend of these investigations. The coset preserving crossed isomorphisms are characterized among the integral crossed isomorphisms, and both integral and coset preserving crossed isomorphisms are characterized among the (1-1) correspondences. Different criteria are given for a crossed isomorphism to be an ordinary isomorphism. Crossed isomorphisms are constructed which induce a given projectivity between the systems of subgroups of A and B. The groups B are studied which can be obtained from an Abelian group A by means of an integral crossed automorphism; the group B need not be Abelian. *R. Brauer* (Toronto, Ont.).

Levi, F. W. Notes on group theory. I, II. J. Indian Math. Soc. (N.S.) **8**, 1–9 (1944). [MF 11270]

Denote by $E(m)$, for m an integer, the transformation mapping every element in a group G upon its mth power. Clearly $E(0)$ and $E(1)$ are endomorphisms; and if $E(m)$ and $E(m')$ are endomorphisms, so is $E(mm')$. The author shows that with $E(m)$ always $E(1-m)$ is an endomorphism, that the set of all integers m such that $E(m)$ is an endomorphic mapping of G upon an Abelian subgroup consists exactly of all the multiples of a suitable integer m_0, that every $E(nm_0)$ maps G upon part of its centrum, and that the integers m for which $E(m)$ is an endomorphism form complete cosets modulo m_0; G is Abelian if, and only if, $m_0=1$. The number m_0 can be any odd number; it cannot be 2; if it is a prime or double a prime, then $E(m)$ is an endomorphism if, and only if, $m=0, 1$.

If S is a subgroup of G and g an element in G, then denote by $L(g, S)$ the transformation mapping x in S upon $xgx^{-1}g^{-1}$, and by $R(g, S)$ the transformation mapping x in S upon $gxg^{-1}x^{-1}$. The author shows that $L(g, S)$ is a homomorphism whenever $R(g^{-1}, S)$ is a homomorphism, that $L(g, S)$ maps S upon an Abelian group whenever both $L(g, S)$ and $L(g^{-1}, S)$ are homomorphisms, and that, denoting by B_S the set of all the elements g in G such that $L(g, S)$ is a homomorphism, the crosscut of B_S and S is a group and B_S consists of full right hand cosets modulo this subgroup.

R. Baer (Urbana, Ill.).

Jabber, M. A. On S-groups. Bull. Calcutta Math. Soc. **35**, 111–113 (1943). [MF 10738]

Groups in which the commutator operation is associative were called S-groups in a paper by F. W. Levi [J. Indian Math. Soc. (N.S.) **6**, 87–97 (1942); these Rev. **4**, 133]. The author generalizes to any finite number n of generators $a_1, a_2, \cdots, a_n$ the studies of Levi for two generators. All

commutators $A_{ij}=a_ia_ja_i^{-1}a_j^{-1}$ lie in the center, so an element

$$s=a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\prod A_{ij}^{P_{ij}}$$

can be given the $n(n+1)/2$ coordinates $[p_1, p_2, \cdots, p_n; P_{12}, P_{13}, \cdots, P_{n-1,n}]$. The product of an element s_1 with coordinates p_i^1 and P_{ij}^1 and an element s_2 with coordinates p_i^2 and P_{ij}^2 is an element s_1s_2 with coordinates $p_i^1+p_i^2$ and $P_{ij}^1+P_{ij}^2-p_i^1p_i^2$. This law of composition applied to ordered sets of $n(n+1)/2$ integers defines a group. Since each commutator is in the center ($p_i=0$), this group is an S-group, here called the free S-group generated by $a_1, \cdots, a_n$. Every S-group generated by n elements is homomorphic to this free S-group. To investigate the S-groups, the author then writes in a canonical form the relations between elements within the commutator group C and between elements in the quotient group G/C, obtaining a set of invariant integers which describe the group. *J. S. Frame.*

Wintgen, Georg. Zur Darstellungstheorie der Raumgruppen. Math. Ann. **118**, 195–215 (1941). [MF 10701]

A space (or Euclidean) group is an infinite discrete group of Euclidean motions and reflections in 3 space, which leave no point, line or plane invariant. There are 230 different (nonintertransformable by affine transformations) such space groups. The author exhibits the primitive (irreducible) bounded representations of these groups. A simple extension of the familiar regular representation theory of finite groups is employed as a tool. *A. L. Foster.*

Mendelsohn, N. S. A group-theoretic characterization of the general projective collineation group. Proc. Nat. Acad. Sci. U. S. A. **30**, 279–283 (1944). [MF 11043]

In the present paper an abstract group G is defined by means of certain axioms which determine the subgroup structure of G; each element of G is defined to be a projective collineation and the subgroups concerned are called points and lines. It is proved that the group G and the group of projective collineations of the points and lines of G (that is, its group of inner automorphisms) are isomorphic. A consideration of the complete group of automorphisms of G is now relatively easy. By specializing a line and an absolute involution on it the author obtains the Euclidean group as given by Thomsen. *G. de B. Robinson* (Ottawa, Ont.).

Littlewood, D. E. Invariant theory, tensors and group characters. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944). [MF 10092]

The paper contains a report on the connections between the following topics: representations of the full linear groups, Young's quantitative substitutional analysis, characters of the symmetric group, tensors, invariant theory, Aronhold's symbolic method. Of course, these connections are well known [compare, for instance, Weyl, Classical Groups, Princeton University Press, Princeton, N. J., 1939; these Rev. 1, 42], but every author will place emphasis on different aspects according to his tastes. Part I of Littlewood's paper deals with tensors; it leads to a proof of the "fundamental theorem" that every concomitant under the full linear group can be obtained by multiplication and contraction of tensors. It is shown that, so far as elementary applications are concerned, the method of tensors is equivalent to the symbolic method of invariant theory.

The second part of the paper deals with Littlewood's second multiplication of S-functions [compare, for instance, D. E. Littlewood, The Theory of Group Characters and Matrix Representations of Groups, Oxford University Press, New York, 1940, chap. 10; these Rev. 2, 3]. If $A^{(\lambda)}$ denotes the representation of the full linear group belonging to a partition (λ), then for a second partition (μ) the expression $[A^{(\lambda)}]^{(\mu)}$ again is a representation of the full linear group. It can therefore be written as a direct sum $\sum A^{(\nu)}$, where the sum is to be extended over certain partitions (ν), each taken with a certain multiplicity. Then Littlewood sets $\{\lambda\}\otimes\{\mu\}=\sum\{\nu\}$ for the S-functions $\{\lambda\}$, $\{\mu\}$, $\{\nu\}$ belonging to these partitions. A number of properties for the multiplication $\otimes$ is given. The significance for invariant theory is obvious. If, for instance, $\{\lambda\}$ corresponds to the pth induced representation of the full linear group and (μ) corresponds to the qth induced representation, then, for each $\{\nu\}$ which appears in the product, the homogeneous form f of pth degree has a concomitant of degree q in the coefficients of f, and of type (ν). There is a corresponding connection with invariant theory in the general case. A technique is developed which allows the computation of the concomitants themselves, at least when the cases considered are not too complicated.

The third part of the paper deals with applications to concrete problems of invariant theory. For instance, a generating function for perpetuants is obtained. Concomitants are treated in many special cases. The following "theorem of conjugates" is proved. If (λ) is a partition of p and if $\{\lambda\}\otimes\{\mu\}=\sum\{\nu\}$, then for even p we have $\{\tilde{\lambda}\}\otimes\{\mu\}=\sum\{\tilde{\nu}\}$, and for odd p we have $\{\tilde{\lambda}\}\otimes\{\tilde{\mu}\}=\sum\{\tilde{\nu}\}$, where the sign $\tilde{}$ indicates that the conjugate partition is to be taken. *R. Brauer* (Toronto, Ont.).

Malcev, A. Orthogonal and symplectic representations of semi-simple Lie groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 318–321 (1943). [MF 11072]

This note is concerned with the determination of all (isomorphic) representations of semi-simple groups by orthogonal matrices and by symplectic matrices. A number of theorems, stated without proof, make it possible to carry over the Weyl-Cartan theory of unitary representations. Let $O(n)$, $O^+(n)$, $Sp(n)$ be the orthogonal, proper orthogonal and symplectic groups of degree n. If two orthogonal (symplectic) representations of degree n of a connected group are equivalent in the general linear group, they are equivalent in $O(n)$ ($Sp(n)$). The orthogonal (symplectic) representations of degree n of semi-simple groups decompose in $O^+(n)$ ($Sp(n)$) into orthogonally (symplectically) irreducible representations. An orthogonally irreducible orthogonal representation of a semi-simple group is either absolutely irreducible or decomposes into two contragredient absolutely irreducible representations. The same holds for symplectic representations. The sum of two contragredient representations is both orthogonal and symplectic.
P. A. Smith (New York, N. Y.).

Tschebotarëw, N. On representation of Lie groups without measure. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 11–13 (1943). [MF 11183]

The author discusses the question of the existence of an invariant measure for a Lie group of transformations. Having obtained the partial differential system on which depends the existence of an invariant measure, the author proceeds to discuss the problem in the case of a transitive representation. The results he obtains in this way are explicitly or implicitly contained in the results obtained by

André Weil in his book "L'integration dans les groupes topologiques et ses applications" [Hermann et Cie., Paris, 1940; these Rev. 3, 198; cf. section 9 of chap. II of the book]. *C. Chevalley* (Princeton, N. J.).

Albert, A. A. Quasigroups. II. Trans. Amer. Math. Soc. 55, 401–419 (1944). [MF 10500]

[The first part appeared in the same Trans. 54, 507–519 (1943); these Rev. 5, 229.] In this second part the author is particularly interested in the properties of normal subloops (sub-quasigroups with unit element). Various criteria for normal subloops are given; the union and crosscut of normal subloops are normal and the law of isomorphism holds. This leads to the general refinement theorem for composition chains and the theorem of Jordan-Hölder. A finite loop is defined to be solvable when none of the composition (quotient) loops have nontrivial subloops.

As in group theory one can formulate an extension theory for loops based on factor sets, and various general properties of these are obtained, differing in many respects from the corresponding theory in groups. Finally, some properties of finite loops are studied. All loops of order $n \leqq 4$ are groups. The loops of order $n=5$ are determined and some special types of order $n=6$ are also treated. *O. Ore.*

ANALYSIS

★**Smirnov, V. I. A Course in Higher Mathematics. Vol. 4.** OGIZ, Leningrad-Moscow, 1941. 620 pp.

[The Russian title is Kurs Vyčšej Matematiki. Volume I, by Tamarkin and Smirnov, first appeared in 1924; volumes II and III, by Smirnov, appeared in 1933.] Volume III of this work was concerned mainly with the theory of functions of a complex variable and with its applications. The present volume contains partial differential equations and integral equations. The material discussed in the present volume is indispensable not only for mathematicians, both pure and applied, but also for physicists and well educated engineers. The exposition is remarkably clear and contains many details which appear to be new. The list of contents by chapters follows. I. Calculus of variations. II. Integral equations. III. Boundary value problems. §1. Boundary value problems in the case of an ordinary differential equation. §2. Statical problems and stationary processes. §3. Equation of conduction of heat and wave-equations. IV. General theory of partial differential equations. §1. Equations of first order. §2. Equations of higher order. *J. D. Tamarkin* (Providence, R. I.).

Bosanquet, L. S. Note on convexity theorems. J. London Math. Soc. 18, 239–248 (1943). [MF 10879]

Given an integrable function $\phi(x)$, let

$$\Phi_\alpha(x)=(1/\Gamma(\alpha))\int_0^x (x-t)^{\alpha-1}\phi(t)dt, \qquad \alpha>0,$$

be the αth integral of $\phi(x)$. The fundamental result of M. Riesz [Acta Litt. Sci. Szeged 1, 114–126 (1923); cf. also Hardy and Littlewood, Proc. London Math. Soc. (2) 11, 411–478 (1913), in particular, pp. 428 ff.] asserts that, if $0<\gamma<r$, $\sigma \geqq 0$, $\beta \geqq 0$, $\phi(x)=O(x^\sigma)$, $\Phi_r(x)=o(x^\beta)$ as $x\to+\infty$, then

$$\Phi_\gamma(x)=o(x^{[\sigma(r-\gamma)+\beta\gamma]/r})$$

as $x\to+\infty$. This theorem is used by the author to obtain a number of results about the typical means and Cesàro sums of infinite series. Thus, let

$$0<\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty, \quad A(x)=A_0(x)=\textstyle\sum_{\lambda_n\leq x}a_n$$

and let $A_r(x)$ be the rth integral of $A(x)$. Let α be real, $\beta>-1$, $r>0$ and let (i) $a_n=O\{\lambda_n{}^\alpha(\lambda_n-\lambda_{n-1})\}$, (ii) $A_r(x)=o(x^\beta)$ as $x\to+\infty$. Then

$$A_\gamma(x)=o(x^{[\alpha(r-\gamma)+\beta(\gamma+1)]/(r+1)})$$

for $0\leqq\gamma<r$. Parallel results (with extensions to one-sided conditions) are obtained for Cesàro sums. *A. Zygmund.*

Tschebotaröw, N. On a general criterion of the minimax. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 341 (1943). [MF 10756]

Let $\mathbf{x}$ and $\mathbf{a}$ be vectors of dimension n and r, respectively, where $\mathbf{x}$ is from a discrete or connected and $\mathbf{a}$ from a connected manifold. Suppose a function $f(\mathbf{x}, \mathbf{a})$ satisfies the following conditions: (1) f has continuous second derivatives with respect to $\mathbf{a}$; (2) let f_0 be a certain constant and $\mathbf{a}$ arbitrary in the neighborhood of a certain $\mathbf{a}_0$; then the vectors $\mathbf{x}$ for which $f(\mathbf{x}, \mathbf{a})>f_0$ form a compact set. The purpose of the author is to characterize the vector $\mathbf{a}_0$ for which $\max_{\mathbf{x}} f(\mathbf{x}, \mathbf{a})$ is a minimum. Let $\mathbf{a}$ be a particular vector. We denote by $\mathbf{y}$ the vector with components $\partial f/\partial a_i$ and let $\mathbf{x}^{(\nu)}$ be all the vectors for which $f(\mathbf{x}, \mathbf{a}_0)$ becomes minimum. We denote the corresponding vectors by $\mathbf{Y}^{(\nu)}$. If for each r-dimensional vector $\mathbf{c}$ we can find a pair of vectors $\mathbf{Y}^{(\mu)}$ and $\mathbf{Y}^{(\nu)}$ so that the scalar products $\mathbf{cY}^{(\mu)}$ and $\mathbf{cY}^{(\nu)}$ are of opposite sign, then $\max f$ is a minimum for the particular $\mathbf{a}_0$. On the other hand, if there exists a vector $\mathbf{c}$ for which all scalar products are of the same sign, then we have no minimum. Tchebychev's best approximation of continuous functions by polynomials of given degree is an interesting example. *G. Szegö.*

Theory of Sets, Theory of Functions of Real Variables

Lusin, N. Sur les parties de la suite naturelle des nombres entiers. C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 175–178 (1943). [MF 11161]

Let small letters a, b, etc. denote sets of integers and let capital letters A, B, etc. denote sets of sets of integers. Let $a \equiv 0$ mean that a is a finite set and let $a \subset b$ mean that $a-b\equiv 0$. Now A and B are orthogonal if $a \cap b \equiv 0$ for every a in A and every b in B; A and B are separable if there exist h and k such that $h \cap k \equiv 0$ and $a \subset h$, $b \subset k$ for every a in A and every b in B. Theorem of du Bois-Reymond. If A and B are denumerable and orthogonal they are separable. In this note the author uses the axiom of choice to construct non-denumerable sets A and B which are orthogonal but not separable. *L. H. Loomis* (Cambridge, Mass.).

Kurepa, Georg. Über eine Eigenschaft von Systemen linearer wohlgeordneter Mengen. Math. Ann. 118, 578–587 (1942). [MF 10715]

This paper belongs to the research series started by the author's thesis [Ensembles ordonnés et ramifiés, Paris,

1935]. There he proposed the following problem. Let F be an arbitrary nondenumerable system of well-ordered (w.-o.) subsets of the ordered set of the rational numbers; does F then contain a subsystem of the same power such that none of its elements is a segment of another of its elements? The author now answers this question in the affirmative and he also proves an analogous theorem for systems of w.-o. linear sets and even for systems of w.-o. subsets of an arbitrary ordered separable set. However, the analogous question for systems of w.-o. subsets of an arbitrary ordered set M possessing the Suslin property (that is, every system of disjoint, nonempty, open intervals of M is denumerable) remains undecided. A partial result in this case leads to separable sets M. *A. Rosenthal* (Albuquerque, N. M.).

Erdös, P. Some remarks on connected sets. Bull. Amer. Math. Soc. **50**, 442–446 (1944). [MF 10614]

This note answers a few questions about connected sets and also asks a few questions whose answers remain unknown. It is shown that the plane is the sum of c disjoint biconnected sets. A set is constructed such that no finite subset disconnects it and such that any countable dense subset totally disconnects it. *D. Montgomery.*

Yang, Cheng-Ning. On the uniqueness of Young's differentials. Bull. Amer. Math. Soc. **50**, 373–375 (1944). [MF 10601]

For S an n-dimensional set, f a function on S, α a limiting point of S, and V and η vectors, the scalar product $V\cdot\eta$ is said to be a Young's differential if

$$\lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x)-f(\alpha)-V\cdot(x-\alpha)}{\|x-\alpha\|}=0.$$

Uniqueness is proved for Young's differential at α if S has n linearly independent limiting directions at α. A vector l is a limiting direction of S at α if $\|l\|=1$ and

$$\text{lower bound}_{x \text{ in } S,\, \delta>\|x-\alpha\|\to 0} \left\| \frac{x-\alpha}{\|x-\alpha\|}-l \right\| =0$$

for all positive δ. *J. F. Randolph* (Oberlin, Ohio).

Cesari, Lamberto. Sulle trasformazioni continue. Ann. Mat. Pura Appl. (4) **21**, 157–188 (1942). [MF 10509]

The author shows that, if the functions $x=x(u,v)$, $y=y(u,v)$ are absolutely continuous in Tonelli's sense and have their first partial derivatives x_u, x_v, y_u, y_v integrable $L^{2+\alpha}$ ($\alpha>0$) on a region of the u,v-plane, then these functions x, y are of limited variation and are absolutely continuous in the sense of Banach and Vitali. It is shown by counterexamples that, if the integrability conditions $L^{2+\alpha}$ ($\alpha>0$) are replaced by the conditions L^2 or L^1, the functions x, y are not necessarily of limited variation or absolutely continuous in the Banach-Vitali sense. *H. H. Goldstine.*

***McShane, Edward James. Integration.** Princeton University Press, Princeton, N. J., 1944. viii+392 pp. $6.00.

The author states in the preface: "This book is written with the hope that it will open a path to the Lebesgue theory which can be traveled by students of little maturity." The first fifty pages gives the descriptive properties of sets in q-dimensional space R_q and properties of real valued functions defined on R_q, such as upper and lower semi-continuity, absolute continuity and bounded variation. The starting point of the definition of integral is the idea of volume of an interval denoted by ΔI. Two properties of this function ΔI are used. (a) For every interval I, ΔI is finite and nonnegative, and is the same as $\Delta\bar{I}$. (b) If a closed interval I is the union of two closed nonoverlapping intervals I_1, I_2, then $\Delta I=\Delta I_1+\Delta I_2$.

Let $I=I_1+I_2+\cdots+I_n$, I_n an interval, $I_jI_k=0$, $j\neq k$, and let $s(x)=c_k$ on I_k. The integral of the step-function $s(x)$ is then defined by

$$\int_I' s(x)dx=\sum_{k=1}^{n} c_k\Delta I_k.$$

Let $f(x)$ be bounded on I, and let upper and lower integrals of $f(x)$ be defined by

$$R\overline{\int_I} f(x)dx=\inf \int_I' s(x)dx, \qquad s(x)\geqq f(x),$$

$$R\underline{\int_I} f(x)dx=\sup \int_I' s(x)dx, \qquad s(x)\leqq f(x).$$

If these upper and lower integrals are equal their common value is the Riemann integral $R\int_I f(x)dx$. The Riemann integral of a continuous function $\varphi(x)$ is called the "elementary integral" or "Cauchy integral" of $\varphi(x)$, denoted by $\int_I\varphi(x)dx$.

The function $u(x)$ on the closed interval I is a u-function if it is (a) lower semi-continuous, (b) bounded above; $l(x)$ is an L-function if it is (a) upper semi-continuous, (b) bounded below. The integrals of $u(x)$ and $l(x)$ are defined by

$$\int_I u(x)dx=\sup \int_I \varphi(x)dx, \qquad \varphi(x)\leqq u(x),$$

$$\int_I l(x)dx=\inf \int_I \varphi(x)dx, \qquad \varphi(x)\geqq l(x).$$

The upper and lower integrals of $f(x)$ on I are then given by

$$\overline{\int_I} f(x)dx=\inf \int_I u(x)dx, \qquad u(x)\geqq f(x),$$

$$\underline{\int_I} f(x)dx=\sup \int_I l(x)dx, \qquad l(x)\leqq f(x).$$

If these upper and lower integrals of $f(x)$ are equal then $f(x)$ is summable, and their common value is the Lebesgue integral $\int_I f(x)dx$ of $f(x)$ over I. If E is any set, $E\subset I$, I closed, $K_E(x)=1$, x on E, $K_E(x)=0$ elsewhere (the characteristic function of E), then

$$\int_E f(x)dx=\int_I f(x)K_E(x)dx.$$

If E is unbounded let W_n be the interval $-n\leqq x^{(i)}\leqq n$, $i=1,\cdots,q$; the integral of $f(x)$ over E is then defined by

$$\int_E f(x)dx=\lim \int_{EW_n} f(x)dx,$$

provided this limit exists.

Thus, in the first one hundred pages, the author develops the theory of the Lebesgue integral without mentioning the concept of measure. How does he introduce this concept? The set E is measurable if for each interval W_n the function $f(x)=1$ is summable over EW_n. The measure of E is given by

$$mE=\lim \int_{EW_n} 1dx.$$

If for real a the set $E(f\geqq a)$ is measurable then $f(x)$ is

measurable. The author now proceeds to develop all the usual results involving measurable sets, measurable functions and summable functions, including the inequalities of Hölder, Schwarz and Minkowski.

This brings us to the end of chapter III, and space permits us to do little more than indicate the scope of the remainder of the book. Chapter IV: reduction of multiple integrals to repeated integrals, integrals as functions of sets, convergence almost everywhere, almost uniform convergence and convergence in measure. Chapter V: properties of the Dini derivates, derivatives of functions which are integrals, mean value theorems for integrals, the necessary and sufficient condition that $F(x)$ be the integral of its derivative and theorems on differentiation under the integral sign. Chapter VI: the continuity properties of measurable functions, the relation between measurable functions and functions of Baire class not greater than 2, metric density of sets, approximate continuity, approximation in the mean of functions of class L_p, the Riemann-Lebesgue theorem, nonmeasurable sets and functions. Chapter VII: the Lebesgue-Stieltjes integral. This is based on a theory of measure m_g with respect to a function $g(x)$ defined on R_q. Associated with $g(x)$ is a function of intervals $\Delta_g I=g(x)|_\alpha^\beta$ which for $q=1$ is $g(\beta)-g(\alpha)$ and for $q=2$ is $g(\beta^{(1)}, \beta^{(2)})-g(\alpha^{(1)}, \beta^{(2)})-g(\beta^{(1)}, \alpha^{(2)})+g(\alpha^{(1)}, \alpha^{(2)})$. This is extended to a q-dimensional interval, and on this function of intervals the author bases his definitions of monotonicity and bounded variation, positively monotonic if, for every closed subinterval I of the closed interval I, $\Delta_g I\geqq 0$, negatively monotonic if $\Delta_g I\leqq 0$. This function of intervals $\Delta_g I$ then replaces ΔI in the theory of the Lebesgue integral to give the Lebesgue-Stieltjes theory of an integral of $f(x)$ with respect to $g(x)$, and a theory of measure m_g with respect to $g(x)$, $g(x)$ of bounded variation. The role of Borel sets in this theory is indicated, and consideration is given to general measure functions. Chapter VIII: the Perron integral; for this the author uses a tetrad of functions φ^1, φ^2, φ^3, φ^4 associated with $f(x)$ for which $-\infty\neq D_+\varphi^1\geqq f$, $+\infty\neq D^-\varphi^2\leqq f$, $-\infty\neq D_-\varphi^3\geqq f$, $+\infty\neq D^+\varphi^4\leqq f$. This lengthens the discussion somewhat, but it enables the author to obtain the formula for integration by parts; the only other known way of obtaining this formula for Perron integrals is first to prove it for the Denjoy integral, and then use the equivalence of the Denjoy and Perron integrals. There is no mention made of Denjoy integrals. Chapter IX: theorems concerning the existence and uniqueness of solutions of a system of differential equations. Chapter X: Vitali's covering theorem for sets in R_q is first proved, and this is used to obtain the usual theorems concerning derivatives of functions of sets and of multiple integrals.

R. L. Jeffery (Kingston, Ont.).

Pi Calleja, Pedro. On the concept of the integral. Revista Soc. Cubana Ci. Fís. Mat. 1, 19–27, 58–62 (1942); 88–91, 123–127 (1943); 164–169 (1944). (Spanish) [MF 7193]

Pi Calleja, Pedro. On the Stieltjes integral. Publ. Inst. Mat. Univ. Nac. Litoral 5, 27 pp. (1943). (Spanish) [MF 10662]

The author considers in detail the equation

$$\int_a^b f(x)d\phi(x)=\int_a^b f(x)\phi'(x)dx,$$

where $\phi(x)$ is of bounded variation and has a finite derivative everywhere. He discusses the problem of determining conditions under which the existence of one of the two integrals appearing in the equation implies that of the other and their equality. The integrals are assumed to be in turn Riemann, Lebesgue, Riemann-Stieltjes, Lebesgue-Stieltjes, in all meaningful combinations. The author's purpose is to give these results, which are essentially known, a unified and in certain cases new presentation. *E. R. Lorch.*

Nöbeling, Georg. Über die Flächenmasse im Euklidischen Raum. Math. Ann. 118, 687–701 (1943). [MF 10721]

It is known that there are compact zero dimensional sets in Euclidean three dimensional space E_3 for which the two dimensional measures under various definitions may differ. In the present note it is shown that even for surfaces these measures may differ, the expedient being to prove the following theorem. Each compact zero dimensional set P of E_3 is contained in a surface F of E_3 such that $F-P$ is the countable sum of triangles the sum of whose areas is finite.

The two dimensional measure functions μ_J, μ_G, μ_C, μ_H and μ_P of Janzen, Gross, Carathéodory, Hausdorff and Favard are also redefined and axioms are given that distinguish the class of two dimensional measure functions of Kolmogoroff. Surfaces are constructed having, respectively, $\mu_J<\mu_G$, $\mu_P<\mu_J$, $\mu_P<\mu_G$, $\mu_J=\mu_G=\mu_P<(\mu_C, \mu_H$, or any Kolmogoroff measure), $\mu_C<\mu_H<\infty$ and it is shown that there are surfaces for which different Kolmogoroff measures do not agree. *J. F. Randolph* (Oberlin, Ohio).

Federer, Herbert. Surface area. I. Trans. Amer. Math. Soc. 55, 420–437 (1944). [MF 10501]

A measure Φ is defined over E_3 which can be thought of as a generalization of Carathéodory linear measure over E_2. If $f(x)$ is continuous over a rectangle R of E_2,

$$S=E_3E_x[(x_1, x_2)\varepsilon R,\ x_3=f(x_1, x_2)],$$

then $\Phi(S)$ equals the Lebesgue area of the surface defined above R. Now $f^*(S)$ is the set $y=f(x)$, $x\varepsilon S$; $P_c(y)$ is the projection of y on a plane whose normal has direction components $c=(c_1, c_2, c_3)$, $|c|=1$, $y\varepsilon E_3$. For $S\subset E_3$, let $\gamma(S)$ be the supremum of $|P^*(S)|$, the Lebesgue plane measure of the set $P^*(S)$. For $A\subset E_3$, $\gamma_r(A)$ is the supremum of numbers of the form $\sum\gamma(S)$, $S\varepsilon F$, F a denumerable family of open connected subsets of E_3 covering A with the diameter of S less than r. Then $\Phi(A)$ is the limit of $\gamma_r(A)$ as r tends to 0^+.

If a set A is Φ-measurable according to the definition of measurability of Carathéodory then for $\epsilon>0$ there is a closed set $C\subset A$ with $\Phi(A-C)<\epsilon$. Also, if $A\subset E_3$, $c\subset E_3$, $|c|=1$, then $\Phi(A)\geqq|P_c^*(A)|$. If f is a function of domain A and range B, ϕ a general measure function over B, F a family of subsets of $T\subset A$, then $V_F(f, T, \phi)$ is the supremum of numbers of the form $\sum\phi[f^*(S)]$, $S\varepsilon G$, $G\subset F$, G a partition of T. Let $p(S)$ be the number of elements in S when S is finite; otherwise $p(S)=\infty$. Finally,

$$N(f, S, y)=p\{S\cdot E_x[f(x)=y]\}.$$

If F covers A in the Vitali sense, if there is a sequence G of partitions of A such that $G_n\subset F$, $n=1, 2, \cdots$, and the supremum of the diameter of S tends to zero as $n\to\infty$, $S\varepsilon G_n$, then

$$\int N(f, A, y)d\varphi(y)=V_F(f, A, \varphi)=\lim_{n\to\infty}\left\{\sum_{S\varepsilon G_n}\varphi[f^*(S)]\right\}.$$

If B is E_2, ϕ is Carathéodory linear measure over B, f is continuous on $[a, b]$ to B, $T_a^b f$ the total variation of f, then

$$\int N\{f, [a, b], y\}d\varphi y = T_a^b f.$$

If $A \subset E_3$, A of class F_σ, $a \varepsilon E_3$, $|a| = 1$, then

$$\Phi(A) \geqq \int N(P_a, A, y)dy.$$

A substantial part of the paper is taken up with the proof of the fact that the Φ-measure of a surface is equal to the Lebesgue area. *R. L. Jeffery* (Kingston, Ont.).

Federer, Herbert. Surface area. II. Trans. Amer. Math. Soc. **55**, 438–456 (1944). [MF 10502]

[Cf. the preceding review.] The problem considered is the validity of

$$\int_{\xi} N(f, T, y)d\Phi y = \int_T Jf(x)dx,$$

f a function on E_m to E_n, $m \leqq n$, $T \subset E_m$, the integrand on the right being the Jacobian associated with f by means of its approximate differential. In this paper P is a projection function on E_n to E_m if and only if there is an orthogonal transformation R on E_n to E_n such that the relations $R(x) = y$ and $P(x) = (y_1, y_2, \cdots, y_m)$ are equivalent whenever x and y are points in E_n. The set $P^*(S)$, $S \subset E_n$, is then used as in part I to define a measure function $\Phi(S)$, $|P^*(S)|$, the m-dimensional Lebesgue measure, replacing plane Lebesgue measure.

If f is a Lebesgue measurable function on E_m to E_n, T is a Lebesgue measurable subset of E_m and if there are sets $T_1, T_2, \cdots$ and numbers $M_1, M_2, \cdots$ such that $T = \sum T_j$ and $|f(z) - f(x)| \leqq M_j|z - x|$ whenever $z \varepsilon T_j$, $x \subset T_j$, then

$$(1) \qquad \int N(f, T, y)d\Phi y = \int_T Jf(x)dx.$$

If f is a Lebesgue measurable function on E_m to E_n, T a Lebesgue measurable subset of E_m, and if, for every $x \varepsilon T$, as $z \to x$, $\limsup |f(z) - f(x)|/|z - x| < \infty$, then (1) holds.

If f is a Lebesgue measurable function on E_2 to E_n $(2 \leqq n)$, T is a Lebesgue measurable subset of E_2 and corresponding to each $x \varepsilon T$ there are three distinct points z^1, z^2, z^3 of E_2 such that $|z^j| = 1$ and $\limsup |\{f(x + tz^j) - f(x)\}/t| < \infty$, as $t \to 0$, for $j = 1, 2, 3$, then (1) holds. If f is a Lebesgue measurable function on E_1 to E_1, T is a Lebesgue measurable set and $|D^+f(x)| < \infty$ for $x \varepsilon T$, then (1) holds with dy replacing $d\Phi(y)$ on the left and $|D^+f(x)|$ replacing $Jf(x)$ on the right. *R. L. Jeffery* (Kingston, Ont.).

Huskey, Harry D. Further contributions to the problem of Geöcze. Duke Math. J. **11**, 333–339 (1944). [MF 10671]

Let $A(S)$ denote the Lebesgue area of a surface S and let $A^*(S)$ denote the area obtained by requiring in the definition of Lebesgue area that all the polyhedra mentioned are inscribed in S. Clearly $A^*(S) \geqq A(S)$ so that the equality holds if $A(S) = +\infty$. The problem of Geöcze is that of determining under what other conditions $A^*(S) = A(S)$.

Let S be of the form $S: z = f(x, y)$, where $f(x, y)$ is continuous on the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Suppose $f(x, y)$ to be extended over the whole space by successive reflections. Let

$$I(x, y; h, k) = \left\{1 + \left[\frac{f(x+h, y) - f(x, y)}{h}\right]^2 + \left[\frac{f(x, y+k) - f(x, y)}{k}\right]^2\right\}^{\frac{1}{2}}.$$

The author proves the following result. If there exist two sequences of positive integers $\{m_i\}$ and $\{n_i\}$ such that

$$\lim_{i \to \infty} \int_0^1 \int_0^1 I(x, y; 1/m_i, 1/n_i)dxdx = \lim_{i \to \infty} \int_0^1 \int_0^1 I(x, y; -1/m_i, -1/n_i)dxdy = A(S),$$

then $A^*(S) = A(S)$. He points out that this includes all previous results on this problem. *C. B. Morrey, Jr.*

Theory of Series

Bradshaw, J. W. More modified series. Amer. Math. Monthly **51**, 389–391 (1944). [MF 11145]

Continuation of an article in Amer. Math. Monthly **46**, 486–492 (1939); these Rev. **1**, 50.

Hadwiger, H. Über die Umordnungsstärke und eine Erweiterung des Steinitzschen Satzes. Math. Ann. **118**, 702–717 (1943). [MF 10722]

Let $\sum v_k$ be a series of vectors, in s-dimensional space E_s, for which $\sum v_k = 0$ and $\sum |v_k| = \infty$. Suppose $\beta > 1$ exists such that $\sum |v_k|^\beta < \infty$, and let λ be the greatest lower bound of such numbers β. To each rearrangement $\sum v_k'$ of $\sum v_k$, and each integer n correspond a greatest integer $N = N(n)$ and a least integer $M(n)$ such that the terms $v_1', \cdots, v_n'$ include $v_1, \cdots, v_N$ and are included among $v_1, \cdots, v_M$. Let

$$x = \limsup_{n \to \infty} \{\log [1 + M(n) - N(n)]\}/\{\log [2 + N(n)]\}.$$

For each $\alpha > 0$, let U_α denote the set of rearrangements $\sum v_n'$ of $\sum v_n$ for which $x < \alpha$. If $\alpha \leqq 1$, the vectors representing values of convergent series in U_α constitute a linear manifold in E_s. If $\alpha \leqq \lambda^{-1}$, this manifold consists of the null vector. *R. P. Agnew* (Ithaca, N. Y.).

* **Szász, Otto. Introduction to the Theory of Divergent Series.** Department of Mathematics, Graduate School of Arts and Sciences, University of Cincinnati, Cincinnati, Ohio, 1944. v+72 pp. $1.25.

Lithoprinted lectures given by Szász in an introductory course in summability and transcribed by J. Barlaz. A novel feature of the course is that the simple regular transformation Y defined by $\sigma_n = \frac{1}{2}(s_{n-1} + s_n)$ is systematically used to illustrate points in the theory. This transformation was previously used by W. A. Hurwitz [Bull. Amer. Math. Soc. **32**, 77–82 (1926)] to illustrate Tauberian theorems. The following result, which the reviewer has not seen elsewhere, is obtained. The series $1 - 2^{-s} + 3^{-s} - \cdots$, which converges to $(1 - 2^{1-s})\zeta(s)$ when $\Re(s) > 0$, is summable Y when $\Re(s) > -1$, and, moreover, for each integer $r > 0$, the series is summable Y^r when $\Re(s) > -r$. It is an interesting fact that methods of summability so simple as Y and its powers can furnish the analytic continuation of $\zeta(s)$ from the region of convergence of $\sum n^{-s}$ to the entire complex plane except $s = 1$. *R. P. Agnew* (Ithaca, N. Y.).

Agnew, Ralph Palmer. Summability of subsequences. Bull. Amer. Math. Soc. **50**, 596–598 (1944). [MF 10854]

The author establishes the following theorem involving the regular method of summability A with infinite matrix of reference of real or complex constants. Let A be regular and let x_n be a bounded complex sequence. Then there exists a subsequence y_n of x_n such that the set L_Y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n. *H. L. Garabedian.*

Agnew, R. P. and Hill, J. D. Summability of bounded sequences. Duke Math. J. **11**, 573–574 (1944). [MF 11090]

Let B denote the space of bounded sequences $x \equiv \{x_k\}$, with the distance (x, y) between two points x and y of B defined as the least upper bound for $k=1, 2, 3, \cdots$ of $|x_k - y_k|$. Let A be a regular matrix method of summability, and let $B(A)$ denote the set of sequences in B which are summable A. The authors give two proofs that in general the set $B(A)$ is not separable. The following theorem is involved in the second demonstration. For each $r>0$ almost all sequences of 0's and 1's are summable by the Cesàro means of order r to the value $\frac{1}{2}$. *H. L. Garabedian.*

Agnew, Ralph Palmer. Euler transformations. Amer. J. Math. **66**, 313–338 (1944). [MF 10575]

The Euler transformation $E(r)$ of order r of a series $u_0+u_1+\cdots$ and its sequence $s_0, s_1, \cdots$ of partial sums may be written in the form

$$\sigma_n = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} s_k.$$

Many writers have developed various properties of the Euler transformation for the case that r is a real number, $0<r<1$. This paper constitutes a thorough study of fundamental properties of the Euler transformation for the general case in which r is complex.

Included among the topics and questions considered are commutative properties of the transformations $E(r)$, inclusion relationships between transformations $E(r)$ and other methods of summability, a necessary condition for summability $E(r)$, consequences arising from the consideration of $E(r)$ summability of the sequence $\{z^k\}$, omission and adjunction of elements, inclusion of $E(r)$ by the generalized Abel method, consistency of the transformations $E(r)$, series-to-series transformations, $E(r)$ summability of power series and a characterization of the open sets in which power series are summable. Finally, it is proved that, if $r_1, r_2, \cdots$ is a sequence of real numbers for which $r_n \to 0$ and $nr_n \to \infty$, then the transformation

$$\sigma_n = \sum_{k=0}^{n} \binom{n}{k} r_n^k (1-r_n)^{n-k} s_k$$

includes $E(r)$ for each $r>0$. The question is raised as to how the $E(r_n)$ transformations are related to each other and to other methods of summability. *H. L. Garabedian.*

Agnew, Ralph Palmer. On sequences with vanishing even or odd differences. Amer. J. Math. **66**, 339–340 (1944). [MF 10576]

As an application of theorems concerning Euler's transformation proved in the paper reviewed above, the author arrives at the following results. Let $x_0, x_1, x_2, \cdots$ be a bounded sequence for which all higher differences $\Delta^n x_0$ with even n vanish; then $x_n = 0$ for all n. If the same condition is satisfied for all odd n then $x_n = x_0$ for all n. *G. Szegö.*

Wang, Fu-Traing. Some remarks on oscillating series. Quart. J. Math., Oxford Ser. **15**, 1–6 (1944). [MF 10682]

Let r be a positive integer and let σ_n denote the C_r (Cesàro) transform of a series $\sum a_n$ with partial sums s_n. If $0<\delta<1$, $n^\delta a_n > -K$ and $n^{r(1-\delta)}(\sigma_n - s) \to 0$, then $\sum a_n$ converges to s. By means of Fourier series, it is shown that there is a sense in which this result cannot be improved. If $0<\delta<\delta'<r^{-1}$, then there exists a divergent series $\sum a_n$ such that $n^\delta |a_n| < K$ and $n^{r(1-\delta')}\sigma_n \to 0$. *R. P. Agnew.*

Fuchs, W. H. J. A theorem on finite differences with an application to the theory of Hausdorff summability. Proc. Cambridge Philos. Soc. **40**, 189–197 (1944). [MF 10787] The page nr. was taken from page proof. In the published issue they are 189–196.

The main result of this paper is embodied in the theorem which follows. Let $a_1 < a_2 < \cdots$ be positive integers, $n(R)$ the number of a's not exceeding R, s_n a sequence of complex numbers. If (i) $n(R) \geqq \frac{1}{2}R$ for $R > R_0$, (ii) $s_n = o(n^K)$ as n tends to infinity, (iii) $\Delta^p s_0 = 0$ $(p = a_1, a_2, \cdots)$, then $s_n = P(n)$, where $P(x)$ is a polynomial of degree less than K. The proof of the main theorem involves some results concerning the closure of sets of functions of the form $e^{-cx}x^{a_\nu}$ which are of intrinsic interest. Finally, the main theorem is applied with respect to two particular Hausdorff methods of summation to yield a curious result. *H. L. Garabedian.*

Minakshisundaram, S. A new summation process. Math. Student **11**, 21–27 (1943). [MF 10994]

Let $\mu > -1$ and let

$$\varphi_\mu(t) = \mu! 2^\mu t^{-\mu} J_\mu(t), \qquad t>0,$$

where $J_\mu(t)$ is Bessel's function of the first kind and of order μ; then $\varphi_\mu(t) \to 1$ as $t \to 0$. Let $\lambda_1, \lambda_2, \cdots$ be an increasing divergent sequence of positive numbers, and let r be a nonnegative integer. A series $u_1+u_2+\cdots$ is summable (J_μ, λ_n, r) to σ if the series in

$$\sigma(t) \equiv \sigma(\mu; \lambda_n; r; t) = \sum_{k=0}^{\infty} \{\varphi_\mu(\lambda_k t)\}^r u_k$$

converges over some interval $0<t<\delta$ to a transform $\sigma(t)$ such that $\sigma(t) \to \sigma$ as $t \to 0+$. These methods of summability are generalizations of the methods (J_μ, n, r) treated by Chandrasekharan in three papers [J. Indian Math. Soc. (N.S.) **6**, 168–180 (1942); Proc. Indian Acad. Sci., Sect. A. **17**, 219–229 (1943); Bull. Calcutta Math. Soc. **34**, 187–196 (1942); see these Rev. **5**, 63, 64]. The method (J_μ, λ_n, r) is regular when $r(\mu+\frac{1}{2})>1$. Necessary conditions for summability and Tauberian theorems are obtained for the methods $(J_\mu, \lambda_n, 1)$. *R. P. Agnew* (Ithaca, N. Y.).

Hardy, G. H. and Littlewood, J. E. Notes on the theory of series. XXII. On the Tauberian theorem for Borel summability. J. London Math. Soc. **18**, 194–200 (1943). [MF 10872]

[The last two notes of this series, published in Proc. London Math. Soc. (2) **41**, 257–270 (1936) and Quart. J. Math., Oxford Ser. **31**, 161–172 (1937), were both numbered XX.] A new proof, simpler than those previously given, of the authors' Borel-Tauber theorem: if $\sum a_n$ is summable by Borel's exponential (or integral) method and $n^{\frac{1}{2}}a_n$ is bounded, then $\sum a_n$ converges. The proof is based on several Tauberian lemmas, involving Borel and other methods of summability, and a theorem of Vitali.

R. P. Agnew (Ithaca, N. Y.).

Cowling, V. F., Leighton, Walter and Thron, W. J. Twin convergence regions for continued fractions. Bull. Amer. Math. Soc. **50**, 351–357 (1944). [MF 10598]

The main result of this paper is embodied in the following theorem. Let k be any real number greater than 1 and let ϵ and ϵ_1 be positive numbers less than k. If the elements $a_n = r_n e^{i\theta_n}$ are such that $r_{2n} \geqq 2[k+\epsilon_1-\cos\theta_{2n}]$, $r_{2n-1} \leqq k-\epsilon$ $(n=1, 2, \cdots)$, the continued fraction

$$1+\frac{a_1|}{|1}+\frac{a_2|}{|1}+\cdots$$

converges. Several corollaries to the main theorem are given.

H. L. Garabedian (Evanston, Ill.).

Fourier Series and Generalizations, Integral Transforms

Minakshisundaram, S. A note on the theory of Fourier series. Proc. Nat. Inst. Sci. India **10**, 205–215 (1944). [MF 11021]

The purpose of the paper is to generalize certain results of Zygmund and Szász on the relation between the jump of a function and its Fourier coefficients. These generalizations are related to the behavior of integrals of the type

$$\phi_\alpha(t) = (1/\Gamma(\alpha)) \int_0^t (t-u)^{\alpha-1}\phi(u)du,$$

$$\phi_\alpha^*(t) = \int_0^t |\phi_\alpha(u)|\,du$$

and to methods of summability of an order depending on α.

R. Salem (Cambridge, Mass.).

Salem, R. Sets of uniqueness and sets of multiplicity. II. Trans. Amer. Math. Soc. **56**, 32–49 (1944). [MF 10789]

The author investigates the "uniqueness" and "multiplicity" character of certain "unsymmetric" perfect sets on $(0, 2\pi)$ [for the terminology, see the first part of the paper [same Trans. **54**, 218–228 (1943); these Rev. **5**, 3], where the "symmetric" case is solved completely]. We quote the main result of the paper which clearly shows that the general situation in the theory of uniqueness of trigonometric series is rather complex, and that a proper key will not be easy to find. Given an interval $I=(a, b)$ of length $L=b-a$, let us consider $d+1$ "white" subintervals, each of length $L\xi$, $0<\xi<1/(d+1)$. The origin of the first and the end-point of the last coincide with a and b, respectively. The distances of the origins of the "white" intervals from a will be denoted by $L\alpha_0=0, L\alpha_1, \cdots, L\alpha_d$. The numbers $d, \xi, \alpha_1, \alpha_2, \cdots, \alpha_d$ are fixed once and for all, and we consider the operation D of removing from I the interiors of the d "black" intervals that separate the "white" ones. We first apply D to the interval $(0, 2\pi)$, then to each of the remaining "white" intervals, and so on. What is left is a perfect set P of measure 0. It turns out that P is a set of uniqueness if and only if the following conditions are satisfied: (i) $\theta=1/\xi$ is a "Pisot-Vijayaraghavan" number, that is, an algebraic integer whose conjugates are all inside the unit circle, (ii) $\alpha_1, \alpha_2, \cdots, \alpha_d$ are algebraic numbers of the field $K(\theta)$.

A. Zygmund (South Hadley, Mass.).

Salem, R. On a theorem of Bohr and Pál. Bull. Amer. Math. Soc. **50**, 579–580 (1944). [MF 10850]

A remarkably simple proof of the theorem of Pál and Bohr [H. Bohr, Acta Litt. Sci. Szeged **7**, 129–135 (1935)] asserting that, if $f(x)$ is any continuous function of period 2π, then there is a strictly increasing continuous function $x=\phi(t)$, $0\leqq t\leqq 2\pi$, satisfying $\phi(0)=0$, $\phi(2\pi)=2\pi$, and such that the Fourier series of $f(\phi(t))$ converges uniformly.

A. Zygmund (South Hadley, Mass.).

Menchoff, D. Sur les sommes partielles des séries trigonométriques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 51–53 (1943). [MF 11052]

(i) There is a continuous function $f(x)$ of period 2π and such that the sequence $\{S_n(x; f)\}$ of the partial sums of the Fourier series of f contains no subsequence which would converge everywhere. (ii) Every continuous and periodic f can be decomposed into a sum f_1+f_2 of two continuous and periodic functions such that both $\{S_n(x; f_1)\}$ and $\{S_n(x; f_2)\}$ contain uniformly convergent subsequences (depending on f). Part (i) is given without proof; a proof of (ii) is sketched.

A. Zygmund (South Hadley, Mass.).

Hardy, G. H. and Littlewood, J. E. Note on the theory of series. XXIII. On the partial sums of Fourier series. Proc. Cambridge Philos. Soc. **40**, 103–107 (1944). [MF 10780]

[The last note of the series appeared in J. London Math. Soc. **18**, 194–200 (1943); these Rev. **6**, 46.] An elementary proof of the fact that, if $f\epsilon L$, if $s_n(x)$ is the nth partial sum of the Fourier series of f and if

$$s^*(x) = \sup\{|s_n(x)|/(\log n)^{\frac{1}{2}}\}_{n=2,3,\cdots},$$

then $s^*(x)\epsilon L^2$ and

$$\int_0^{2\pi} \{s^*(x)\}^2 dx \leqq A\int_0^{2\pi} f^2(x)dx,$$

where A is an absolute constant. This is a special case of a general theorem of Littlewood and Paley [Proc. London Math. Soc. (2) **43**, 105–126 (1937)] pertaining to classes L^p, $1<p\leqq 2$, but the proof of the general result is much more difficult.

A. Zygmund (South Hadley, Mass.).

Nachbin, Leopoldo. Some theorems on series with positive terms with application to the generalization of a theorem of Fatou on the absolute convergence of Fourier series. Math. Notae **4**, 90–104 (1944). (Spanish) [MF 11025]

The purpose of the paper is to show that the well-known Fatou theorems on absolute convergence of trigonometric series can be considered as consequences of results of the following type. If $\lambda_n \geqq 0$, $\sum\lambda_n=\infty$, $a_n \geqq 0$, a_n monotonic decreasing, the convergence of $\sum\lambda_n a_n$ implies the convergence of $\sum a_n$ for all sequences $\{a_n\}$ if and only if

$$\underline{\lim}\,(\lambda_1+\cdots+\lambda_n)/n>0.$$

Other results of the same type are given.

R. Salem.

Szász, Otto. On uniform convergence of Fourier series. Bull. Amer. Math. Soc. **50**, 587–595 (1944). [MF 10853]

A sequence of functions $s_n(t)$ defined in the neighborhood of a point τ is said to converge uniformly at that point if (a) $s_n(\tau)$ converges as $n\to\infty$, (b) $s_n(\tau)-s_n(t_n)\to 0$ for any sequence $\{t_n\}\to\tau$. The author proves the following results. (i) If

$$\phi(t) \sim \tfrac{1}{2}a_0+\sum_1^\infty a_\nu \cos \nu t$$

is continuous at $t=0$ and satisfies the condition

$$(*) \qquad \lim_{\lambda\to 1}\limsup_{n\to\infty}\sum_{\nu}^{n\lambda} a_\nu^{+}=0, \qquad a^{+}=\max(0,a),$$

then $\sum a_\nu \cos \nu t$ converges uniformly at $t=0$ [in the case $a_\nu=O(1/\nu)$, see Rogosinski, Math. Z. **41**, 75–136 (1936)]. (ii) If

$$\psi(t)\sim\sum a_\nu \sin \nu t$$

is continuous for $t=0$ and satisfies (*), then $\sum_1^n \nu a_\nu=o(n)$, and $\sum a_\nu \sin \nu t$ converges uniformly at $t=0$. (iii) If $\psi(t)\to\pi A/2$ as $t\to+0$ and satisfies (*), then $\sum_1^n \nu a_\nu\sim An$, and

$$\sum_1^n a_\nu \sin \nu t_n - A\sum_1^n \nu^{-1}\sin \nu t_n\to 0$$

as $t_n\to 0$. *A. Zygmund* (South Hadley, Mass.).

Wang, Fu Traing. A convergence criterion for a Fourier series. Duke Math. J. **11**, 435–439 (1944). [MF 11080]

Given the Fourier series $f(t)\sim\frac{1}{2}a_0+\sum^{\infty}A_n$, where $A_n=a_n\cos nx+b_n\sin nx$, and writing

$$\phi(t)=\tfrac{1}{2}[f(x+t)+f(x-t)-2s],$$

$$\phi_\beta(t)=(1/\Gamma(\beta))\int_0^t (t-u)^{\beta-1}\phi(u)\,du, \qquad \beta>0,$$

the author has shown that, if $\phi_\beta(t)=o(t^\gamma)$ $(\gamma>\beta)$ and if $A_n>-Kn^{-\beta/\gamma}$, then the Fourier series converges to the sum s at the point x [Proc. London Math. Soc. (2) **47**, 308–325 (1942); J. London Math. Soc. **17**, 98–107 (1942); these Rev. **4**, 37]. The purpose of the present paper is to show that the result holds if the condition $\phi_\beta(t)=o(t^\gamma)$ is replaced by $\phi_\beta(t)=O(t^\gamma)$, provided $\int_0^t|\phi(u)|\,du=o(t)$ as $t\to 0$, the other condition remaining the same. *R. Salem.*

Wang, Fu Traing. A note on strong summability of Fourier series. Anais Acad. Brasil. Ci. **16**, 149–152 (1944). [MF 11007]

Hardy and Littlewood have proved [Fund. Math. **25**, 162–189 (1935)] that, at every point x_0 where the periodic and L-integrable function $f(x)$ satisfies

$$(*) \qquad \int_0^t|\varphi(u)|\,du=o(t),$$

$$\varphi(u)=\tfrac{1}{2}\{f(x_0+u)+f(x_0-u)-2f(x_0)\},$$

the partial sums $s_n(x_0)$ of the Fourier series of f have the property

$$\sum_{\nu=1}^{n}[s_\nu(x_0)-f(x)]^2=o(n\log n).$$

Following the very familiar pattern, the author replaces the condition (*) by

$$\int_0^t|\varphi(u)|\,du=O(t), \qquad \int_0^t\varphi(u)\,du=o(t).$$

A. Zygmund (South Hadley, Mass.).

Feng, Nai-Chian. The Cesàro summability of the conjugate series of a Fourier series. Duke Math. J. **11**, 451–458 (1944). [MF 11082]

Let $f(x)$ be of period 2π, integrable L, and let $\psi(t)=f(x+t)-f(x-t)$,

$$\Delta_\eta^{(m)}\psi(t)=\sum_{\nu=0}^{m}(-1)^{m+\nu}\binom{m}{\nu}\psi(t+\nu\eta).$$

If the integral

$$g(t)=(1/2\pi)\int^{\pi}\psi(t)\cot\tfrac{1}{2}t\,dt=\lim_{\epsilon\to+0}\int_{\epsilon}^{\pi}$$

exists, and if

$$\lim_{h\to\infty}\overline{\lim_{\eta\to 0}}\,\eta^{\rho}\int_{h\eta}^{\pi}|\Delta_\eta^{(m)}\psi(t)|\,t^{-1-\rho}\,dt=0$$

for some positive ρ, then the series conjugate to the Fourier series of f is summable (C,ρ) at the point x to sum $g(x)$.

A. Zygmund (South Hadley, Mass.).

Lorch, Lee. The Lebesgue constants for Borel summability. Duke Math. J. **11**, 459–467 (1944). [MF 11083]

By a well-known theorem of C. N. Moore, there exist continuous functions whose Fourier series are not summable by Borel's method at some points. The proof depends on the unboundedness of a certain integral analogous to the integral defining Lebesgue's constants. The author makes a study of this integral and gives its asymptotic value.

R. Salem (Cambridge, Mass.).

Natanson, I. P. On the convergence of trigonometrical interpolation at equidistant knots. Ann. of Math. (2) **45**, 457–471 (1944). [MF 10921]

It is a familiar fact that, while there is a certain analogy between the convergence of trigonometric interpolation (with equidistant fundamental points) and the convergence of Fourier series, the integral tests of the latter are usually not valid in interpolation. This, as the author shows, applies, in particular, to the Dini-Lipschitz integral test. If, however,

$$|f(x_0+t)+f(x_0-t)-2f(x)|\leqq\omega(t), \qquad 0<t\leqq\pi,$$

where $\omega(t)$ is nondecreasing and $\int_0^\pi t^{-1}\omega(t)\,dt$ finite, the interpolating polynomials of the periodic function $f(x)$ converge to $f(x_0)$ at the point x_0. *A. Zygmund.*

Pollard, Harry. The mean convergence of non-harmonic series. Bull. Amer. Math. Soc. **50**, 583–586 (1944). [MF 10852]

The main purpose of the paper is to prove that, if the sequence $\{\lambda_n\}$ satisfies

$$\left(\sum_{-\infty}^{+\infty}|\lambda_n-n|^a\right)^{1/a}=\delta<\pi^{-1}\log 2,$$

where $a=2p/|2-p|$, then any function $f(x)\in L^p$ $(p>1)$ admits a unique representation

$$\text{l. i. m.}_{\substack{(p)\\ N\to\infty}}\sum_{-N}^{N}c_n e^{i\lambda_n x}.$$

For $p=2$, the result coincides with a result of Duffin and Eachus [same Bull. **48**, 850–855 (1942); these Rev. **4**, 97].

R. Salem (Cambridge, Mass.).

Pollard, Harry. Fourier series with coefficients in a Banach space. Bull. Amer. Math. Soc. **50**, 581–582 (1944). [MF 10851]

Let $f(t)$ be in $(0,1)$ a function whose values are elements of a complex Banach space B. Let $f(t)\in L^2$ in the sense of Bochner and let $c_n=\int_0^1 e^{2\pi i n t}f(t)\,dt$. The purpose of the paper is to prove that Bessel's inequality (which has been shown by Bochner not to be valid for certain spaces B) is valid only if B is "unitary." *R. Salem* (Cambridge, Mass.).

Bellman, Richard. Almost orthogonal series. Bull. Amer. Math. Soc. **50**, 517–519 (1944). [MF 10843]

Let $\phi_n(x)$ be a set of functions of class $L^2(a,b)$ such that $\int_a^b|\phi_n|^2dx=1$. Define $a_{mn}=\int_a^b\phi_m\phi_n\,dx$, $\delta^2=\sum_{m\neq n}|a_{mn}|^2$.

The author is concerned with establishing analogues of Bessel's inequality and the Riesz-Fischer theorem under the hypothesis $\delta^2<\infty$. The following theorems are established. (1) If $f\varepsilon L^2(a, b)$ and $b_n=\int_a^b f\phi_n dx$, then

$$\sum|b_k|^2\leqq(1+\delta)\int_a^b|f|^2dx.$$

(2) If $\sum|b_k|^2<\infty$, there exists a function $f\varepsilon L^2(a, b)$ such that

$$\sum_1^\infty\left|b_k-\int_a^b f\bar{\phi}dx\right|^2\leqq\delta^2\sum|b_k|^2.$$

For $\delta=0$ these reduce to the usual theorems. Theorem (1) is a special case of a result of Boas which gives a condition both necessary and sufficient for the validity of Bessel's inequality [Amer. J. Math. 63, 361–370 (1941); these Rev. 2, 281]. *H. Pollard* (New York, N. Y.).

Romanov, N. P. On an orthogonal system. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 257–258 (1943). [MF 11180]

If $\theta_n(x)=nx-[nx]$ (on (0, 1)) and $\psi_n(x)$ is the corresponding orthonormal set, the author gives the formula

$$\psi_n(x)=(\sqrt{12}/\sqrt{\varphi_2(n)})\sum\mu(n/d)\theta_d(x)d,$$

where the summation is over all divisors d of n and

$$\varphi_2(n)=\sum\mu(n/d)d^2.$$

He extends this result to other functions involving the Bernoulli polynomials, proves completeness and from the Parseval equality on the corresponding $\psi_n(x)$ obtains some identities concerning primes (which he does not give in this paper). No proofs are given. *P. Erdös.*

Tscherkassoff, A. Functions with complete systems of powers. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 245–249 (1943). (Russian. English summary) [MF 10834]

A system of functions $\phi_i(x_1, x_2, \cdots, x_m)$, $i=1, 2, \cdots, k$, defined on a closed domain G of an m-dimensional Euclidean space is said to possess a complete system of powers if every continuous function $f(x_1, x_2, \cdots, x_m)$ on G can be approximated with arbitrary degree of accuracy by polynomials of the form

$$\sum a^{\mu_1\mu_2\cdots\mu_k}\phi_1^{\mu_1}\phi_2^{\mu_2}\cdots\phi_k^{\mu_k}$$

with integral $\mu_1, \mu_2, \cdots, \mu_k$. The author shows that the necessary and sufficient condition for this is that for any u_i the system of equations

$$\phi_i(x_1, x_2, \cdots, x_m)=u_i, \qquad i=1, 2, \cdots, k,$$

has at most one solution $x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}$. This necessarily implies $k\leqq m$. The proof depends on Weierstrass' well-known approximation theorem. The author also gives a sufficient condition for two functions $\phi_1(x)$, $\phi_2(x)$ defined on $a\leqq x\leqq b$ to possess a complete system of powers in the case when these functions have continuous first derivatives on $a\leqq x\leqq b$. *A. C. Offord* (Newcastle-on-Tyne).

Segal, I. E. The span of the translations of a function in a Lebesgue space. Proc. Nat. Acad. Sci. U. S. A. **30**, 165–169 (1944). [MF 10795]

If $f(x)$ belongs to L_p in $(-\infty, \infty)$, then the linear combinations $\sum c_n f(x+\delta_n)$ will sometimes be dense in L_p. For $p=1$ the necessary and sufficient condition is that the Fourier transform of $f(x)$ shall be strictly nonvanishing. For $p=2$ it is that the Fourier transforms shall vanish on a Lebesgue zero set at most. The crucial question left open was whether the last condition is also sufficient for $1<p<2$. The author proves that it is not. His main tool is the lemma that, for any $q>2$, there exists in $(-\infty, \infty)$ a function of bounded variation, purely irregular and continuous, with (variational) spectrum of measure zero, such that its Fourier-Stieltjes transform is an element of L_q. This lemma follows in turn from a construction by R. Salem of such a function of bounded variation in $(0, 2\pi)$ whose Fourier-Stieltjes coefficient is $O(n^{-1+\epsilon})$, for any fixed $\epsilon>0$ given in advance. *S. Bochner* (Princeton, N. J.).

Pollard, Harry. The Bernstein-Widder theorem on completely monotonic functions. Duke Math. J. **11**, 427–430 (1944). [MF 11078]

A short and self-contained proof of the theorem that a completely monotonic function in $(0, \infty)$ can be represented as a Laplace integral [see Widder, The Laplace Transform, Princeton University Press, Princeton, N. J., 1941, pp. 160 ff.; these Rev. 3, 232]. *R. P. Boas, Jr.*

Delange, Hubert. Une nouvelle démonstration de certains théorèmes taubériens. C. R. Acad. Sci. Paris **217**, 309–311 (1943). [MF 11098]

Let K be a function increasing for $x>0$; let $\psi(t)$ be a non-decreasing function such that $\psi(0)=0$ and $\int_0^\infty K(t/x)d\psi(t)$ exists for $x>0$; let $\rho(x)$ be a differentiable function such that $x\rho'(x)\log x$ approaches zero as $x\to\infty$ $(x\to 0)$; H is a positive constant. The author states (without proof) three sets of conditions under which the relation

$$\int_0^\infty K(t/x)d\psi(t)\sim Hx^{\rho(x)}, \qquad x\to\infty\ (x\to 0),$$

is equivalent to

$$\psi(t)\sim Ht^{\rho(t)}/c[\rho(t)], \qquad t\to\infty\ (t\to 0),$$

where

$$c(\lambda)=\int_0^\infty K(u)d(u^\lambda).$$

As an interesting application the following result is mentioned. Let $p(u)$ be the sum of the reciprocal of the primes less than u. Then $p(u)\sim\log\log u$. *H. Pollard.*

Fraser, W. C. G. The factorial transform. Duke Math. J. **11**, 469–486 (1944). [MF 11084]

The factorial transform is defined by the integral

$$f(s)=\int_0^\infty\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}d\alpha(t),$$

where $\alpha(t)$ is of bounded variation in every finite interval. For suitable choice of $\alpha(t)$ as a step-function this becomes the familiar factorial series $\sum a_n B(s, n+1)$. Such integrals can be rewritten as Laplace integrals and share many properties with them. The author studies questions of convergence, uniqueness and analytic continuation in a manner analogous to Widder's study of the Laplace transform. The question of inversion is reserved for a later paper.
H. Pollard (New York, N. Y.).

Reed, I. S. The Mellin type of double integral. Duke Math. J. **11**, 565–572 (1944). [MF 11089]

The usual inversion and Parseval theorems for Mellin transforms are generalized to double integrals. As an application, an interpolation formula of Ramanujan [Hardy,

"Ramanujan," Cambridge University Press, Cambridge, England, 1940, p. 186, formula (A); these Rev. **3**, 71] is generalized to functions of two variables and applied to the evaluation of some integrals containing hypergeometric functions. *R. P. Boas, Jr.* (Cambridge, Mass.).

Gupta, H. C. Operational calculus and Hankel transforms. J. Indian Math. Soc. (N.S.) **7**, 117–128 (1943). [MF 10927]

Pairs of functions which are Hankel transforms of each other are derived from the well-known relation [Tricomi, Accad. Naz. Lincei. Rend. **22**, 564–571 (1935)] between the Laplace transforms of a function and its Hankel transform, and from an analogous relation, quoted without proof from a paper by the author [communicated for publication], between the inverse Laplace transforms of a function and its Hankel transform. Beside elementary functions, Bessel functions, hypergeometric functions and integral sine and cosine appear in the functions considered. Not all the examples listed are claimed to be new. *A. Erdélyi.*

de Oliveira Castro, F. M. The operational calculus in the complex domain. Anais Acad. Brasil. Ci. **16**, 59–72 (1944). [MF 10893]

Some of the rules of the operational calculus are extended to the complex domain. These are derived by a method which makes evident the algebraic structure of the problem at hand. *A. E. Heins* (Cambridge, Mass.).

NUMERICAL AND GRAPHICAL METHODS

***Peters, J. Eight-Place Table of Trigonometric Functions for Every Sexagesimal Second of the Quadrant.** Edwards Brothers, Inc., Ann Arbor, Michigan, 1943. xi+901 pp. $20.00.

The original was published by the German Office of Surveying (Reichsamt für Landesaufnahme), Berlin, 1939.

Fletcher, Alan. Note on tables of an integral. Philos. Mag. (7) **35**, 16–17 (1944). [MF 10863]

The "Tables of an Integral" by E. O. Powell [Philos. Mag. (7) **34**, 600–607 (1943); these Rev. **5**, 110] are compared and checked with certain related tables by Spence and by Newman. *W. E. Milne* (Corvallis, Ore.).

Terrill, H. M. and Sweeny, Lucile. An extension of Dawson's table of the integral of e^{x^2}. J. Franklin Inst. **237**, 495–497 (1944). [MF 11146]

Terrill, H. M. and Sweeny, Lucile. Table of the integral of e^{x^2}. J. Franklin Inst. **238**, 220–222 (1944). [MF 11147]

In the first paper $\int_0^x e^{t^2}\,dt$ is tabulated for $2 \leqq x \leqq 4$, in the second for $0 \leqq x \leqq 2$. The tabular interval is .01; the number of decimal places decreases from six at the beginning to two at the end, but the number of significant figures increases from five to nine. *W. Feller* (Providence, R. I.).

Stäblein, F. und Schläfer, R. Numerische Berechnung von $y(x)=e^{-x^2}\int_0^x e^{t^2}dt$. Z. Angew. Math. Mech. **23**, 59–61 (1943). [MF 9785]

Values, to four decimal places, of $y(x)$ are tabulated for $0<x<10$ in steps of .1. The computation is based on a series expansion derived from the differential equation $y'=-2xy+1$. *W. Feller* (Providence, R. I.).

Comrie, L. J. Recent progress in scientific computing. J. Sci. Instruments **21**, 129–135 (1944). [MF 10998]

Lecture before the Institute of Physics and the London Mathematical Society.

Hussain, S. T. A method of extracting the nth root of a positive number. Math. Student **11**, 12–15 (1943). [MF 10992]

Richmond, H. W. On the Newton-Raphson method of approximation. Edinburgh Math. Notes no. **34**, 5–8 (1944). [MF 11302]

Hitchcock, Frank L. An improvement on the G.C.D. method for complex roots. J. Math. Phys. Mass. Inst. Tech. **23**, 69–74 (1944). [MF 10628]

A contribution to the determination of complex roots of equations with real coefficients by seeking approximate real quadratic factors of the polynomial. Let $P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0=0$ be the equation, x^2+px+q a trial factor, $P(x)=Q(x)\ (x^2+px+q)+(r_1x+r_0)$. Dividing again, we obtain $Q(x)=R(x)\ (x^2+px+q)+(u_1x+u_0)$. Let $v_1=u_0-pu_1$, $v_0=-qu_1$ and $p'=(r_1u_0-r_0u_1)/(v_1u_0-v_0u_1)$ and finally $q'=(-r_1v_0+r_0v_1)/(v_1u_0-v_0u_1)$. If x^2+px+q is a sufficiently close approximation to a quadratic factor of $P(x)$, then $x^2+(p+p')x+(q+q')$ will be a better approximation.

As an example, the author discusses an eighth degree equation. The successive trial divisors are x^2+x+1, $x^2+.95x+.86$, $x^2+.94303x+.86047$ and $x^2+.9430166365x+.8604737372$. It is stated that the tenth decimal place is in doubt.

The example worked seems to show that the method may in a few steps give a very good approximation to the factors and hence to the roots. However, no automatic check on the degree of accuracy obtained is furnished. Unless the first trial factor is a sufficiently close approximation to an actual factor, the method will not converge. A final paragraph gives hints on the finding of a satisfactory first trial divisor, but their practical usefulness is limited.

A. J. Kempner (Boulder, Colo.).

Synge, J. L. A geometrical interpretation of the relaxation method. Quart. Appl. Math. **2**, 87–89 (1944). [MF 10339]

For the meaning of the term "relaxation method," compare Southwell's book [Relaxation Methods in Engineering Science, Oxford University Press, New York, 1940; these Rev. **3**, 152]. The following equations are to be solved:

$$\sum_{j=1}^{n} a_{ij}x_j - b_i = 0, \qquad i=1, \cdots, n,$$

$a_{ij}=a_{ji}$ and $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ positive definite. The paper gives a geometrical interpretation of the relaxation method, which leads the author to suggest a modification which promises to accelerate the approach of the approximations toward the required limit. *A. J. Kempner* (Boulder, Colo.).

Spoerl, Charles A. On solving simultaneous linear equations. Trans. Actuar. Soc. Amer. **45**, 18–32, discussion, 67–69 (1944). [MF 10936]

A sequel to the author's paper in the same Trans. **44**, 276–288 (1943); these Rev. **5**, 161.

Lonseth, A. T. On relative errors in systems of linear equations. Ann. Math. Statistics **15**, 323–325 (1944). [MF 11328]

A system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = c_i, \qquad i=1, 2, \cdots, n,$$

is considered involving n unknowns x_j and $n(n+1)$ coefficients a_{ij}, c_i. A criterion is found for allowable relative errors in the coefficients to assure nonvanishing of the determinant of the a_{ij}'s, a bound is found for the relative error in each unknown, and a bound is found for the relative errors of the coefficients sufficient to insure a preassigned accuracy in the unknowns. The article contains some typographical errors. In equation (1), the summation is from $j=1$, not $i=1$; and in relation (6) ρ and $|\rho_{ik}|$ should be interchanged.

T. E. Sterne (Aberdeen, Md.).

Ullman, Joseph. The probability of convergence of an iterative process of inverting a matrix. Ann. Math. Statistics **15**, 205–213 (1944). [MF 10827]

Let A be the matrix to be inverted, C_0 a first approximation to A^{-1}. Hotelling's second approximation is $C_1=C_0(2-AC_0)$ [same Ann. **14**, 1–34 (1943); these Rev. **4**, 202]. The author studies this method of iteration and also the alternative

$$C_1=C_0\{1+(1-AC_0)+(1-AC_0)^2\},$$

which is occasionally more efficient. For a given C_0 the question arises whether C_1 will be a better approximation or, conversely, how good an approximation C_0 ought to be in order to make the probability of convergence reasonable. The author finds quite satisfactory estimates for the probability in question, assuming that the elements of C_0 are independent random variables whose mean values are the corresponding elements of A^{-1}. The distribution functions are assumed to be either of the rectangular type (as will be the case if the inaccuracy is due only to rounding errors) or of the Gaussian type (which assumption is justified by practical considerations). *W. Feller* (Providence, R. I.).

Semendiaev, K. A. The determination of latent roots and invariant manifolds of matrices by means of iterations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 193–222 (1943). (Russian. English summary) [MF 10076]

An extensive study of the evaluation of characteristic roots and vectors of finite matrices by iteration processes. Special attention is paid to the case of nonlinear elementary divisors. In the first part the author studies the evaluation of the dominant root and the corresponding characteristic vectors. This section is in part a new exposition and in part a continuation of Aitken's studies [Proc. Roy. Soc. Edinburgh. Sect. A. **57**, 269–304 (1937)]. New is the author's treatment of the higher characteristic values, where he completely discards the usual procedure of "deflation." Instead, he transforms the given matrix A into the form $A^*=HAH^{-1}$, where H is a matrix such that at least one row and one column of A^* contain only zeros outside the main diagonal. This, of course, amounts to a reduction of the order of A. The matrices H and H^{-1} are explicitly determined by means of the characteristic vectors belonging to the dominant root. The transform A^* of a symmetric matrix is not necessarily symmetric. *W. Feller.*

Tucker, Ledyard R. The determination of successive principal components without computation of tables of residual correlation coefficients. Psychometrika **9**, 149–153 (1944). [MF 11096]

The usual procedure of computing the successive characteristic values of symmetric matrices consists in a process of deflation [Hotelling, Psychometrika **1**, 27–35 (1936); Aitken, Proc. Roy. Soc. Edinburgh. Sect. A. **57**, 269–304 (1937)]. The author proposes instead to border the given matrix A with a new row and a new column, each of the form $(i\lambda x_1, i\lambda x_2, \cdots, i\lambda x_n, -\lambda)$, where λ is the dominant characteristic value of A, $(x_1, \cdots, x_n)$ the corresponding normed characteristic vector and $i^2=-1$. The new matrix has the same characteristic values except for λ which is replaced by zero. The bordering is, of course, much simpler than deflation; however, the subsequent iterative process becomes more troublesome. *W. Feller.*

Idelson, N. On the computation of weights of the unknowns in the method of least squares. Astr. J. Soviet Union **20**, 11–13 (1943). (Russian. English summary) [MF 10748]

If $a_{ij}x_j=b_i$ are the normal equations in a problem of least squares the weight of x_j is the reciprocal of the coefficient of b_j in the expression for x_j in terms of the b's. The author obtains these coefficients by putting in n columns to the right of the b's with $1, 0, 0, \cdots$ in the first column, $0, 1, 0, \cdots$ in the second, etc. The process of solving the normal equations by successive eliminations is extended to these added columns and the desired coefficients are obtained along with the values of the x's. The calculation is arranged to avoid the labor of finding unwanted coefficients (for example, the coefficient of b_k in the expression for x_j). *W. E. Milne.*

Richter, Hans. Zur Ausgleichung der Beobachtungsreihen. Monatsh. Math. Phys. **50**, 14–26 (1941). [MF 10488]

"Es werde ein Versuch angestellt, dessen Ergebnis eine Funktion $f(x)$ ist, . . . von der man . . . nur weiss, dass sie einer linearen Differentialgleichung k. Ordnung genügt, deren Koeffizientenfunktionen man . . . erst durch den Versuch bestimmen will. Man weiss also . . .

$$(1) \qquad f(x)=f_0^*(x)+\sum_{\nu=1}^{k}\lambda_\nu f_\nu^*(x)$$

mit konstanten λ_ν. . . . Der n-malig angestellte Versuch . . . liefere . . . $f_1(x), \cdots, f_n(x)$, von denen . . . höchstens $k+1$ linear unabhängig sein dürften, die jedoch in Wirklichkeit . . . unabhängig . . . sein werden. Es ergibt sich . . . die Aufgabe, zu den $f_\nu(x)$ neue Funktionen $g_\nu(x)$ so zu finden, dass . . . die $g_\nu(x)$ eine möglichst gute Annäherung an die $f_\nu(x)$ bilden und dass . . . die $g_\nu(x)$ eine Abhängigkeit voneinander der durch (1) gegebenen Form haben; als möglichst gute Annäherung wird man dabei eine solche im Sinne der Methode der kleinsten Quadrate, gegebenenfalls noch unter Verwendung von Gewichten, verstehen . . . Es soll weiter . . . beachtet werden, dass die Beobachtungsresultate meist . . . als diskontinuierliche Beobachtungsreihen äquidistanter Argumentstellen gegeben sind: eine Beobachtungsreihe wird daher zweckmässiger durch einen Vektor $\mathfrak{f}_\nu$ . . . charakterisiert . . . Die Aufgabe nimmt nunmehr die folgende Gestalt an, die wir Approximation k. Ordnung nennen wollen. Gegeben sind die n Vektoren $\mathfrak{f}_1, \cdots, \mathfrak{f}_n$ mit den Gewichten $\lambda_1^2, \cdots, \lambda_n^2$. Gesucht sind (a) im homogenen Fall n Vektoren $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$, (b) im inhomogenen Fall $n+1$ Vektoren $\mathfrak{g}_0, \mathfrak{g}_1, \cdots, \mathfrak{g}_n$

derart, dass unter den $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$ k linear unabhängig sind und

$$\sum_{\nu=1}^{n} \lambda_\nu{}^2(\mathfrak{f}_\nu - \mathfrak{g}_\nu)^2, \text{ resp. } \sum_{\nu=1}^{n} \lambda_\nu{}^2(\mathfrak{f}_\nu - \mathfrak{g}_0 - \mathfrak{g}_\nu)^2$$

zum Minimum wird."

The nonhomogeneous case is reduced by a straightforward argument to the homogeneous case. The approximation of order k is shown to be equivalent to k successive approximations of order 1. These are furnished by theorem 4, as follows. "Sind $f_1(x), \cdots, f_n(x)$ die beobachteten Funktionen, $\mathfrak{d} = (d_1, \cdots, d_n)$ der Eigenvektor mit $\mathfrak{d}^2 = 1$, der zum grössten Eigenwert der Matrix mit den Elementen $f_{\nu\mu} = \int_a^b f_\nu(x) f_\mu(x) dx$ gehört, dann liefern $g_\nu(x) = d_\nu \sum_\lambda d_\lambda f_\lambda(x)$ die beste homogene Approximation erster Ordnung."

A. J. Kempner (Boulder, Colo.).

Levenberg, Kenneth. A method for the solution of certain non-linear problems in least squares. Quart. Appl. Math. **2**, 164–168 (1944). [MF 10809]

In order to solve a set of n nonlinear equations involving fewer than n unknowns, it is usual to determine differential corrections to a trial solution by a least squares solution of n "linearized" equations involving partial derivatives of the first order only. It is frequently found that the differential corrections so determined lead to a solution that yields a larger sum of squares than the trial solution, because of "overshooting." In this paper a method is developed for diminishing or preventing the overshooting. Instead of minimizing the sum of squares S of the residuals of the linearized equations, one minimizes the quantity

$$wS + a(\Delta\alpha)^2 + b(\Delta\beta)^2 + c(\Delta\gamma)^2 \cdots,$$

where $w, a, b, c, \cdots$ are positive constants and where $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$, etc. are the differential corrections.

T. E. Sterne (Aberdeen, Md.).

Curry, Haskell B. The method of steepest descent for non-linear minimization problems. Quart. Appl. Math. **2**, 258–261 (1944). [MF 11133]

The problem is to find a stationary point of a function $G(x_1, \cdots, x_n)$ of n real variables. A stationary point may be approximated numerically by starting at any point $(x_1', x_2', \cdots, x_n')$ and finding the direction in which the surface $y = G(x_1, \cdots, x_n)$ descends most rapidly. One continues in that direction until the descent terminates. Then one stops, takes a new direction of steepest descent, continues, etc. The convergence of this process is discussed, and it is stated that Cauchy devised the method in 1847 but gave no proof of convergence. *T. E. Sterne.*

Peach, M. O. Simplified technique for constructing orthonormal functions. Bull. Amer. Math. Soc. **50**, 556–564 (1944). [MF 10848]

A sequence of linearly independent complex-valued functions $\{f_n\}$ of one or more real variables over a region R is made orthonormal by replacing each f_n by g_n, where g_n is a linear combination of $f_1, \cdots, f_n$ and where the coefficients of combination are determined by the conditions

$$\int_R g_m \cdot \bar{g}_n dR = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

The familiar Schmidt process of orthonormalization determines g_n recurrently from $g_1, \cdots, g_{n-1}, f_n$. As integration of combinations of the functions $\{f_n\}$, $\{g_n\}$ is involved in the process, it is seen that the Schmidt method would lead to lengthy computations in case these integrations must be performed approximately (either because the functions f_n are given as "tabulated" functions, or because the integrations cannot be effected exactly). The present paper shows how to minimize the manual labor of computation by developing an algorithm that obtains the coefficients in the linear combination

$$g_n = c_{n1} f_1 + \cdots + c_{nn} f_n$$

directly from the functions $f_1, \cdots, f_n$, without intercession of the intermediate functions $g_1, \cdots, g_{n-1}$.

I. M. Sheffer (State College, Pa.).

Rosenhamer, H. Synthese ganzer rationaler Funktionen. Z. Angew. Math. Mech. **22**, 153–162 (1942). [MF 8897]

In connection with the study of certain problems in damped vibrations, the author finds it useful to develop a set of formulas for the coefficients a_j of a real polynomial $g(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ having zeros $x_k = v + iw_k$, $k = 1, 2, \cdots, n$, where v and w_k are real and $w_{2p} = -w_{2p-1}$ for $2 \leqq 2p \leqq n$ and $w_n = 0$ if n is odd. The coefficients are expressed in terms of the powers of v and the elementary symmetric functions of the $w_k{}^2$. The use of the formulas is illustrated by application to a special sixth degree equation and to more general quadratic and cubic equations.

M. Marden (Milwaukee, Wis.).

Spoerl, Charles A. Difference-equation interpolation. Trans. Actuar. Soc. Amer. **45**, 70–82 (1944). [MF 10938]

Discussion of the paper in the same Trans. **44**, 289–325 (1943); these Rev. **5**, 160.

Lowan, Arnold N. and Salzer, Herbert E. Coefficients for interpolation within a square grid in the complex plane. J. Math. Phys. Mass. Inst. Tech. **23**, 156–166 (1944). [MF 11141]

This paper contains formulas and tables for the real and imaginary parts of the coefficients $L_m^{(2)}(P)$ $(m = -1, 0, 1)$ and $L_m^{(3)}(P)$ $(m = -1, 0, 1, 2)$ of approximating quadratic and cubic polynomials $\sum_{m=-1}^{n-1} L_m^{(n)}(P) f(z_m)$, $n = 2, 3$, for a function $f(z)$ whose values are assumed known at the corners z_{-1}, z_0, z_1, z_2 of a square whose sides, of length h, are parallel to the real and imaginary axes. The point z_{-1} corresponds to the upper left corner and the other arguments are associated with the remaining three corners taken in counter-clockwise order. The formulas are given in terms of P and also in terms of p and q, where $P = (z - z_0)/h = p + iq$. The tables give the exact values of the real and imaginary parts of the quadratic (3 point) and cubic (4 point) coefficients for p and q ranging from 0 to 1 at intervals of 0.1.

D. Moskovits (Pittsburgh, Pa.).

Lowan, A. N. and Salzer, H. E. Formulas for complex interpolation. Quart. Appl. Math. **2**, 272–274 (1944). [MF 11137]

This paper contains formulas for the real and imaginary parts of the coefficients $A_k(P)$ of the Lagrange-Hermite interpolation polynomial

$$f(z) = \sum_{k=m}^{m+n} A_k(P) f(z_k)$$

for the important special case when the $(n+1)$ points $z_m, z_{m+1}, \cdots, z_{m+n}$ are equidistantly spaced along any straight line in the complex plane. The formulas cover the cases ranging from complex quadratic (3 points) to complex quintic interpolation (6 points) and are given in terms of p

and q, where $p+iq=P=(z-z_0)/h$; h is the complex interval $z_{n+1}-z_n$, and z_0 is the point $z_{n+n/2}$ or $z_{n+(n-1)/2}$ according as n is even or odd. *D. Moskovitz* (Pittsburgh, Pa.).

Salzer, Herbert E. A new formula for inverse interpolation. Bull. Amer. Math. Soc. **50**, 513–516 (1944). [MF 10842]

This paper gives a formula for inverse interpolation in a table of equally spaced arguments. The value of $x-x_0$ is expanded in a series of powers of $f(x)-f(x_0)$ with coefficients which are rational functions of the tabular entries. By using a set of auxiliary quantities $r, s, t, \cdots$, the author obtains the series in a form independent of the number of points used in the interpolation. The quantities $r, s, t, \cdots$ are quotients of linear combinations of values of f and these forms are dependent on the number of points.

W. E. Milne (Corvallis, Ore.).

Salzer, Herbert E. Coefficients for numerical integration with central differences. Philos. Mag. (7) **35**, 262–264 (1944). [MF 10858]

The author gives a table of the coefficients in the formula for numerical integration with central differences. All coefficients are derived from the well-known representation of Bernoulli's numbers as exact fractions. The coefficients of order 20 and lower are retained in this form whilst the coefficients of order 22 to 56 are given to 18 places of decimals. *H. O. Hartley* (London).

Massera, José L. Formulae for finite differences with applications to the approximate integration of differential equations of first order. Bol. Fac. Ingen. Montevideo **2**, (Año 8), 439–507 (1943). (Spanish) [MF 10838]

Massera, José L. Formulae for finite differences with application to the approximate solution of differential equations of first order. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1**, 1–69 (1943). (Spanish) [MF 10935]

These papers are identical with a paper by the same author previously reviewed [Publ. Inst. Mat. Univ. Nac. Litoral **4**, 99–166 (1943); these Rev. **4**, 283].

Chadaja, F. G. On the error in the numerical integration of ordinary differential equations by the method of finite differences. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 97–108 (1942). (Russian. Georgian summary) [MF 10284]

The nth order differential equation

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$$

with initial conditions $y^{(k)}=y_0^{(k)}$, $k=0, 1, \cdots, n-1$, can be integrated numerically by the use of formulas of the type

$$y_\mu^{(k)}=y_{\mu-1}^{(k)}+\tfrac{1}{2}\sum_{\lambda=1}^{n-k-1}(h^\lambda/\lambda!)[y_{\mu-1}^{(k+\lambda)}+(-1)^{\lambda+1}y_\mu^{(k+\lambda)}]-(-h)^{n-k}\sum_{\lambda=0}^{r}\gamma_{\mu,\lambda,k}\nabla_0^\lambda y^{(n)}+R_\mu^{(k)},$$

in which the γ's are constants. This formula expresses $y_\mu^{(k)}$ at the μth point in terms of the higher derivatives at that and the preceding point together with backward differences up to the rth order of $y^{(n)}$ and the remainder term $R_\mu^{(k)}$, which depends on intermediate values of $y^{(n+r+1)}$ and is of degree $n-k+r+1$ in h. Denoting the error in the computed value of $y_\mu^{(k)}$ by $\epsilon_\mu^{(k)}$ and neglecting terms of higher degree in h than $n-k+r+1$, the author obtains his basic formula

$$\epsilon_\mu^{(k)}=\epsilon_{\mu-1}^{(k)}+\tfrac{1}{2}\sum_{\lambda=1}^{n-k-1}(h^\lambda/\lambda!)[\epsilon_{\mu-1}^{(k+\lambda)}+(-1)^{\lambda+1}\epsilon_\mu^{(k+\lambda)}]+R_\mu^{(k)}.$$

The value of $R_\mu^{(k)}$ can be estimated from the difference table for $y^{(n)}$ and the ϵ's can then be calculated in succession. A similar treatment is given for the method of integration due to Mikeladze. *W. E. Milne* (Corvallis, Ore.).

de la Fuente, Julio. A solution of the Dirichlet problem and its application to torsion. Bol. Fac. Ingen. Montevideo **2**, (Año 6), 112–124 (1942). (Spanish) [MF 10837]

The paper is concerned with a graphical solution of the problem of Dirichlet based on covering the area with a net of curves forming small squares [for a bibliography in connection with the torsion problem: T. J. Higgins, J. Appl. Phys. **14**, 469–480 (1943), in particular, p. 477; these Rev. **5**, 26]. A numerical example concerning a triangular cross section is worked out in the paper. *I. Opatowski.*

Fox, L. Solution by relaxation methods of plane potential problems with mixed boundary conditions. Quart. Appl. Math. **2**, 251–257 (1944). [MF 11132]

The author obtains approximation formulas for the solution of $\nabla^2 w=0$ for the values of w at mesh points near a curved boundary in the plane by a combination of Liebmann's formula and linear interpolation for the two cases: (1) when w is given on the boundary; (2) when the normal derivative $\partial w/\partial n$ is given on the boundary. He applies his formulas and relaxation methods to two problems, namely, to find the function w harmonic in the circle $x^2+y^2-2x-2y+1=0$ and satisfying the boundary conditions:

(1) $$\partial w/\partial n=(y-x)/(x^2+y^2);$$

(2) $$(\partial/\partial n-1/r)w=[y-x-r\tan^{-1}(y/x)]/r^2,$$

where $r=(x^2+y^2)^{\frac{1}{2}}$. *D. Moskovitz* (Pittsburgh, Pa.).

Emmons, Howard W. The numerical solution of partial differential equations. Quart. Appl. Math. **2**, 173–195 (1944). [MF 11127]

This paper contains a detailed expository treatment of Southwell's relaxation process for the numerical solution of partial differential equations. Examples are included illustrating the application of the process to the solution of boundary value problems for the Laplace, Poisson and biharmonic equations, the equation of natural modes of vibration of a membrane, and other equations of more general type. *D. Moskovitz* (Pittsburgh, Pa.).

* **Stumpff, Karl. Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung.** J. W. Edwards, Ann Arbor, Michigan, 1944. vii+174 pp. $5.25.

Reproduction of a book published by J. Springer [Berlin, 1939]. The book is a sequel to the textbook by the same author [Grundlagen und Methoden der Periodenforschung, J. Springer, Berlin, 1937].

Wiseman, Robert T. The use of punched card equipment for the calculation of policy values and guarantees. Trans. Actuar. Soc. Amer. **45**, 83–88 (1944). [MF 10939]

Discussion of the paper in the same Trans. **44**, 326–342 (1943); these Rev. **5**, 162.

* **Ford, Lester R. Alignment charts.** Notre Dame Mathematical Lectures, no. 4, pp. 1–29. University of Notre Dame, Notre Dame, Ind., 1944.

MATHEMATICAL PHYSICS

MacAdam, David L. On the geometry of color space. J. Franklin Inst. **238**, 195–210 (1944). (1 plate) [MF 11013]

The relations between colors and the significance of some recent research on color vision are explained in terms of geometric analogues. A metric is impressed on the conventional affine chromaticity diagram by direct use of recent determinations of standard deviations of visual matching of equiluminous colors. This method, being empirical, differs radically from the classical theories of Helmholtz and Schrödinger. A general method of constructing a model of a surface from known values of the metric coefficients is described. A photograph is shown of a model of the surface thus constructed from the metric coefficients which are based on experimental color discrimination data. This surface has variable curvature, both positive and negative. Experimental data are not yet sufficient for impressing a metric on the three-space representing variable luminance as well as chromaticity, but it appears probable that such a space would be non-Euclidean. The analyses which led other authors [D. E. Spencer, same J. **236**, 293–302 (1943); P. Moon and D. E. Spencer, J. Opt. Soc. Amer. **33**, 260–269 (1943)] to propose a plane metric for chromaticity and a Euclidean metric for combined luminance and chromaticity are discussed briefly in two appendices, the latter containing a purely mathematical discussion which in the author's opinion shows the assumptions of Spencer and Moon to be unjustifiable. *M. Herzberger* (Rochester, N. Y.).

Maier, Wilhelm. Reflexkurven von Röntgenstrahlen an Kristallen. Ann. Physik (5) **40**, 85–120 (1941). [MF 11190]

The table of contents reads as follows. I. Katoptrik der Reflexkurven (Spiren). II. (a) Elliptische, (b) parabolische, (c) hyperbolische Spiren. III. Abhängigkeit der Spiren von der Art und Lage der Netzebenen. IV. Bestimmung der Lage und Art der Netzebenen aus den Spiren. V. Geometrischer Ort der Schnittpunkte der verlängerten Reflexstrahlen. VI. Die Reflexkurven von Röntgenstrahlen an Steinsalz. VII. Veränderliche Reflexstärke bei den Spiren. VIII. Abweichungen von der Spirenform. (Röntgenstrahlenbrechung.) IX. Knotenbildung der Reflexkurven von Röntgenstrahlen an Steinsalz.

Gauster-Filek, Wilhelm. Über Strahlengeometrie in parallelgeschichteten Medien. Akustische Z. **8**, 175–185 (1943). [MF 10934]

An expository treatment of sound propagation in stratified media on the purely geometrical (ray) basis (without going into the limitations of this approximation). The paper is mainly concerned with graphical aids for the treatment of numerical problems, based on the logarithmic velocity diagram. Energy density and power radiation are treated on the same basis. *H. G. Baerwald* (Cleveland, Ohio).

Ambarzumian, V. A. On the problem of diffuse reflection of light. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 323–334 (1943). (Russian) [MF 11224]

Ginsburg, V. L. On the influence of the terrestrial magnetic field on the reflection of radio waves from the ionosphere. Acad. Sci. USSR. J. Phys. **7**, 289–304 (1943). [MF 11001]

First the author reviews the theory of propagation of plane electromagnetic waves through an isotropic ionized medium under presence of a homogeneous magnetic field. The results are well known except for a trebling phenomenon which takes place for small angles α between the propagation and magnetic field vectors; it is due to the split-up of a degeneracy for $\alpha=0$, where "ordinary" and "extraordinary" waves become of equal status. Next the case of an inhomogeneous medium with stratified carrier density is taken up and treated by methods of geometrical optics (modified eikonal equation). Order of magnitude conditions for the limits of validity of this type of approximation are established which are analogous to, though more involved than, those well-known for the isotropic case; in particular, the phenomenon of "reflection" is outside those limits while the domain of treble waves is accessible to this treatment. An approximate calculation of the transmission coefficient D by the variational method of the Ritz type is presented. In this approximation D is found to depend exponentially on a logarithmic gradient of the carrier density at the "reflection point," and it is pointed out that the trebling phenomenon could be used as a "probe" for measuring this gradient by registering vertical multiple reflections of wave pulses in arctic regions. The paper winds up with a qualitative discussion of reflection and polarization, in particular, in the trebling region. *H. G. Baerwald.*

Grosskopf, J. Die Ausbreitung elektromagnetischer Wellen über inhomogenem Boden. Hochfrequenztech. Elektroak. **62**, 103–110 (1943). [MF 10730]

The Sommerfeld formula for wave propagation over a plane homogeneous earth may be used for nonhomogeneous earth by a suitable modification of the definition of Sommerfeld's numerical distance parameter. Thus at zero heights the author writes

$$w_0=\int_0^r \frac{ik}{2\bar{\epsilon}(r)}dr,$$

where $k=2\pi/\lambda$, $\bar{\epsilon}=\epsilon+i(2\sigma/f)$, and either ϵ or σ or both may be varying with distance r. Then the Sommerfeld function

$$F(w_0)=1-2w_0^{\frac{1}{2}}e^{-w_0}\int_{ix}^{w_0^{\frac{1}{2}}} e^{z^2}dx$$

may be used to determine the field components and gives good agreement with experimentally observed values. For small heights a satisfactory approximation is obtained by using the parameter $w=w_0+ikz/\sqrt{\bar{\epsilon}}$. The author discusses the variations in the Zenneck rotating field for variable earth conductivity in the direction of propagation, and the modifications required when the ground constants are also varying in the azimuthal plane. *M. C. Gray.*

Schelkunoff, S. A. On waves in bent pipes. Quart. Appl. Math. **2**, 171–172 (1944). [MF 10811]

Karlem Riess has proved [same Quart. **1**, 328–333 (1944); these Rev. **5**, 220] that no true transverse electric and transverse magnetic type walls can exist in a bent pipe of rectangular section. In this note the author points out that electrically oriented waves do exist in bent rectangular channels; these waves are obtainable from the product of Bessel and trigonometric functions; their magnetic or electric ellipse is normal to the axis of bending. Similar solutions exist for the "pipe" formed by two concentric spheres and coaxial cones whose vertex is at the sphere center.

H. Poritsky (Schenectady, N. Y.).

Müller-Strobel, Josef und Patry, Jean. Die Rahmenantenne als Empfangsantenne. Helvetica Phys. Acta **17**, 159–167 (1944).

An approximate formula for a circular frame aerial as receiver at relatively low frequencies is derived along the lines of Hallén's theory. If ν is the wave number, a the frame radius, ρ the wire radius, then dissipation, all higher powers of $a\nu$ and finally a/ρ are neglected, but $\log(a/\rho)$ is taken into account. The result can be written in the form

$$J(Z_a+Z_r)=15A/i\lambda, \quad Z_a=L[r_0+30i(\ln(8a/\rho)-2)],$$

where J is the antenna current at terminals, Z_a the antenna impedance, Z_r the receiver input impedance, A the total area, that is, the winding area times the number of turns, λ the wavelength, L the total wire length of aerial, r_0 the wave impedance per unit length of aerial.

H. G. Baerwald (Cleveland, Ohio).

Müller-Strobel, Josef und Patry, Jean. Die gerade Empfangsantenne. Ableitung einer Näherungsformel für den Antennenstrom. Helvetica Phys. Acta **17**, 127–132 (1944).

Practical formulas for the receiving effect of a balanced straight thin aerial for frequencies not close to a resonance are developed on the basis of Hallén's theory [cf. the preceding review]. If $\kappa\equiv 2\pi\lambda^{-1}$ (λ the wavelength), $L\equiv\kappa l$ (l the length of the aerial) and if the tangential component E of an incoming wave is represented by an expansion $\sum_{(n)}E_n(\kappa x)^n$ along the aerial $|x|\leqq l$, of which only the terms with E_0, E_1 and E_2 are kept, the antenna current $J(0)$ for the associated receiver impedance Z_r becomes, in Lorentz-Heaviside units,

$$J(0)=(ic/\kappa\Omega)\sum_{n=0}^{2}f_nE_n\cdot[1+g\cdot Z_r]^{-1},$$

provided $\cos L\neq 0$, that is, $l\neq(2n+1)\lambda/4$; here $f_0\equiv\sec L-1+\Omega^{-1}\cdot G_0(L)$, $f_1\equiv L\sec L-\tan L+\Omega^{-1}\cdot G_1(L)$, $f_2\equiv(1/12)L^4+\Omega^{-1}\cdot G_2(L)$, $g\equiv(ic/2\Omega)(\tan L-\Omega^{-1}Q(L))$;

$$Q(L)\equiv\tan L[4M(L)-(2+i\tan L)\cdot M(2L)-2\cdot\ln 2]+i(M(2L)+4M(L)),$$

$$G_0(L)\equiv 4M(\tfrac{1}{2}L)-4\sec L\cdot M(L)+\sec L(1+i\tan L)\cdot M(2L),$$

$$G_1(L)\equiv Q(L)+2\tan L+2\cdot\ln 2\cdot L\sec L-4L\sec L\cdot M(\tfrac{1}{2}L)+(1+i\tan L)L\sec L\cdot M(2L),$$

with $M(x)\equiv\ln(2\gamma x)^{\frac{1}{2}}-\mathrm{Ci}(2x)+i\,\mathrm{Si}(2x)$. Simpler approximations restricted to the range $L\leqq 0.3$ are also given.

H. G. Baerwald (Cleveland, Ohio).

Linnebach, A. Mehrkreisige Siebschaltungen mit ausgeglichener Resonanzkurve. Elektr. Nachr. Techn. **20**, 238–250 (1943). [MF 10965]

Under the assumption that the relative bandwidth $\ll 2$, the vectorial voltage gain of a wave filter consisting of cascaded decoupled (tube-separated) simple antiresonance circuits can be written as a polynomial in the complex frequency variable. If the condition is imposed that the maximum deviation from the ideal filter amplitude characteristic should be minimal, which amounts to requiring that all possible extrema should be real, all maxima and all minima equal (Tchebyshev condition), the polynomials become even-order Tchebyshev polynomials. The connection between number of circuits, gain fluctuation and relative bandwidth is derived and the complete numerical design data are obtained. *H. G. Baerwald* (Cleveland, Ohio).

Whitehead, S. Mathematical methods applicable to linear phenomena. J. Sci. Instruments **21**, 73–80 (1944). [MF 10758]

This paper gives a brief nonrigorous introduction to several mathematical fields useful in electric circuit theory and other linear systems. The subjects touched on include topology, matrix theory, Laplace and Fourier transforms and loss phase relations for physically possible admittances. No new results are obtained. *C. E. Shannon.*

Kron, Gabriel. Equivalent circuit of the field equations of Maxwell. I. Proc. I. R. E. **32**, 289–299 (1944). [MF 11257]

An equivalent circuit is developed to represent the Maxwell field equations for regions containing conductors and bound charges. This circuit enables one to study transient and sinusoidal field phenomena by the network analyzer, numerical, or circuit methods. With such a development one can handle such problems as radiation from antennas, propagation through wave guides, eddy currents in conductors, etc. *A. E. Heins* (Cambridge, Mass.).

Kaplan, Wilfred and Dresden, M. The mechanics of the condensation of gases. Phys. Rev. (2) **66**, 16–20 (1944). [MF 10777]

The central idea of this paper is to relate the process of condensation to that fundamental change which occurs in the nature of the energy hypersurface of a classical Hamiltonian model for a gas when the energy constant passes through the value zero. The gas is regarded as a system of N particles of potential energy

$$V=\sum_{i\neq j}(a/r_{ij}^{n}-b/r_{ij}^{m}), \quad a>0;\ b>0;\ n>m>0,$$

where r_{ij} is the distance between the ith and jth particles. Such a V gives attraction between pairs of distant particles, repulsion between close ones. If $M(c)$ represents the energy hypersurface $T+V=c$ in the $6N$-dimensional space of coordinates and momenta (T=kinetic energy), then it is shown (with the aid of a slight extension of the authors' arguments) that its projection on the $3N$-dimensional coordinate space is bounded or extends to infinity according as $c<0$ or $c\geqq 0$. Since V has a finite minimum, the hypervolume of $M(c)$ will be finite in the former case (and, of course, infinite in the latter).

The thesis is advanced that the change of $M(c)$ as c passes from values not less than zero to values less than zero is the mathematical counterpart of the process of gaseous condensation. The essential idea of the argument can be rendered as follows. When $c<0$, $M(c)$ being bounded, a physical container such as a box centered at the center of gravity of the system (which point is fixed or in uniform motion on account of the nature of V) can contain the system and be so large that no molecule ever encounters the boundary. When $c\geqq 0$, every container will cut $M(c)$, and the elements of ergodic theory show that every particle will have a unit probability of colliding with the boundary. Thus it is appropriate to regard the second case as representing a gas, the first case its condensation.

Rough computations based on natural assumptions are given which provide what the reviewer regards as striking quantitative confirmations of this most important theory in the case of many actual gases.

The reviewer takes exception to the following mathematical views. (1) The authors describe the change in $M(c)$

as a topological one. Actually they never establish that such a change occurs. The issue of the question depends upon the nature of the infinite region in the space (and this is never made clear) and on the way $M(c)$ is cut by the figures surrounding certain singular loci. The mere difference of boundedness is a metrical, not a topological, property. While in most cases there would be a topological change as c passes through zero, there would also be a prodigious number of such changes as the value of c decreases. Why select just one of them to represent condensation? (2) The statement that exceptional loci (of much lower dimensionality) in a space have "only a slight effect on its structure" is false. (3) The authors do not realize that their argument, coupled with the fact that, on account of the nature of V, there is for each $c<0$ a positive minimum distance between any pair of molecules, enables it to be proved that $M(c)$ $(c<0)$ is bounded; all they claim to prove is that it has "infiniteness only in a small set of directions," a proposition not only obscure but not serving as a solid basis for their later arguments. (4) "The effect of the walls (when there are collisions) may be described as one of continuously varying the initial conditions" is impossible for the reviewer to accept.

It is because of both its vagueness and its apparently being based on these questionable points that in writing this review the reviewer has had to paraphrase the leading argument of the paper, maintaining the essential ideas.

B. O. Koopman (Washington, D. C.).

Putman, Henri. The dynamics of a diffusing gas. Quart. Appl. Math. **2**, 267–270 (1944). [MF 11135]

From hydrodynamical considerations Stefan derived an equation for the diffusion of two gases. Stated in words we have that there acts on a particle of the first gas a force due to the pressure gradient of the first gas and a force proportional to the difference of the velocities of the two gases. The author discusses the special case in which the velocity of the second gas is negligible. *A. E. Heins.*

Ledermann, Walter. Asymptotic formulae relating to the physical theory of crystals. Proc. Roy. Soc. London. Ser. A. **182**, 362–377 (1944). [MF 10747]

Part I deals with finite lattice sums (which occur in lattice dynamics) extended over the N particles of a crystal lattice. It is shown that, since N is large, these sums are very nearly equal to certain corresponding triple Fourier series which are absolutely and uniformly convergent. It follows that the boundary particles have only a negligible influence on the density of vibration frequencies provided the critical ratio of the number ν of boundary particles to N is sufficiently small. [A summary of Part II was published in Nature **151**, 197–198 (1943); these Rev. **4**, 206.]

C. C. MacDuffee (Madison, Wis.).

Kalašnikov, S. The influence of thermal movement on the surface scattering of light in crystals. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 295–305 (1943). (Russian) [MF 11221]

Gow, Margaret M. The thermodynamics of crystal lattices. IV. The elastic constants of a face-centered cubic lattice with central forces. Proc. Cambridge Philos. Soc. **40**, 151–166 (1944). [MF 10785]

Landau, L. On the theory of the intermediate state of superconductors. Acad. Sci. USSR J. Phys. **7**, 99–107 (1943). [MF 9966]

The author sketches a theory of the phase structure within a superconductor in the so-called intermediate state. Shape and number of the boundaries between the superconducting and normal regions are determined assuming that in the normal regions the magnetic induction $B=H_e$ (H_c=critical magnetic field), that in the superconducting regions $B=0$, that the free energy difference between the two states $\Delta F=(1/8\pi)H_c^2$ (per cm^3) and that there is a surface energy $(1/8\pi)H_c^2\cdot\Delta$ per cm^2 boundary, where Δ has the dimensions of a small length. The theory is applied to a plate in a transversal magnetic field. The result is a complicated layer structure of the two states, the laminae of the normal state branching towards the surface in an infinite series of finer and finer laminae. [It seems to the reviewer that the basis of the discussion is not consistent with the fact that the magnetic field penetrates to a depth λ into the superconducting state. Actually the existence of λ necessitates a surface energy $(1/8\pi)H_c^2\cdot\Delta$ with $\Delta\geqq\lambda$. The author apparently considers $\lambda\ll\Delta$. On the other hand he is led to assume $\Delta\to 0$ when the temperature approaches the transition point for which case it is known that $\lambda\to\infty$.] *F. W. London.*

Geilikman, B. T. On the theory of ferromagnetism. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 399–410 (1943). (Russian) [MF 11226]

Planck, Max. Versuch einer Synthese zwischen Wellenmechanik und Korpuskularmechanik. II. Ann. Physik (5) **40**, 481–492 (1941). [MF 11191]

The first part appeared in Ann. Physik (5) **37**, 261–277 (1940); these Rev. **1**, 351.

BIBLIOGRAPHICAL NOTE

Universidad Nacional de la Plata. Publicaciones de la Facultad de Ciencias Fisicomatematicas. Serie Segunda. Revista.

The system of numbering peculiar to this journal, and its changes, make the following explanation desirable. (1) The qualification "Serie Segunda" must be considered as synonymous with "Revista." A first and third series appear simultaneously. (2) Every issue carries on the cover page in central position and in heavy type a number; these numbers are consecutive. At first, they were interpreted as volume numbers in the Mathematical Reviews. (3) Since the sixth number (vol. 6 in our original interpretation) the issues carry also a Roman volume number and a number of the issue in the volume. Thus, in the new system, there was no vol. I, while vol. II, núm. 1 still carries the number 6 of the original enumeration, vol. II, núm. 2 carries number 7, etc. (4) In addition to the number denoted by núm., each issue carries in the upper right corner another number, denoted by N°. All three systems of numbering as well as the year are now quoted in Mathematical Reviews.

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